5. Dimension of schemes

Our aim in this section is give a formal definition of the dimension of a variety, to compute the dimension in specific examples and to prove some of the interesting properties of the dimension.

Definition 5.1. *Let X be a topological space.*

*The **dimension of** X is equal to the supremum of the length n of strictly increasing sequences of irreducible closed subsets of X,*

$$\varnothing \neq Z_0 \subset Z_1 \subset \cdots \subset Z_n.$$

*We will call a chain **maximal** if it cannot be extended a longer chain.*

Note that if X is Noetherian then the dimension of X is, by definition, equal to the maximal dimension of an irreducible component. Note that also that the dimension of X is equal to the dimension of any dense open subset, and that the dimension of any subset is at most the dimension of X.

In general this notion of dimension is a little unwieldy, even for Noetherian topological spaces (in fact, it is pretty clear that this definition is useless for any topological space that is not Noetherian or at least close to Noetherian).

For quasi-projective varieties it is much better behaved. For example,

Theorem 5.2. *Let X be a quasi-projective variety.*

Then the dimension of X is equal to the length of any maximal chain of irreducible subvarieties.

Definition 5.3. *Let $f\colon X \longrightarrow I$ be a map from a topological space to an ordered set I. We say that f is **upper semi-continuous**, if for every $a \in I$, the set*

$$\{\, x \in X \mid f(x) \geq a \,\},$$

is closed in X.

The key result is:

Theorem 5.4. *Let $\pi\colon X \longrightarrow Y$ be a dominant morphism of quasi-projective varieties. Then the function*

$$\mu\colon X \longrightarrow \mathbb{N},$$

is upper semi-continuous, where $\mu(p)$ is the local dimension of the fibre $X_p = \pi^{-1}(\pi(p))$ at p. Moreover if X_0 is any irreducible component of X, Y_0 the closure of the image, we have

$$\dim(X_0) = \dim(Y_0) + \mu_0,$$

where μ_0 is the minimum value of μ on X_0.

Note that semi-continuity of μ is equivalent to saying that the dimension can jump up on closed subsets, but not down. For example, consider what happens for the blow up of a point. In this case, μ is equal to zero outside of the exceptional divisor and it jumps up to one on the exceptional divisor.

We will prove these two results in tandem. Let $d = \dim X$. We will need an intermediary result, which is of independent interest:

Lemma 5.5. *Assume* $(5.2)_d$.

If $X \subset \mathbb{P}^n$ *is a closed subset of dimension* d *and* $H \subset \mathbb{P}^n$ *is a hypersurface then*

$$\dim(X \cap H) \geq \dim(X) - 1,$$

with equality if and only if $H \cap X$ *does not contain a component of* X *of maximal dimension.*

Proof. We might as well assume that X is irreducible and that H does not contain a component of X of maximal dimension. Pick a maximal chain of irreducible subvarieties of X which contains a component Y of $X \cap H$,

$$\varnothing \neq Z_0 \subset Z_1 \subset \cdots \subset Z_e.$$

Then $X = Z_e$ and $Y = Z_i$, some i. As we are assuming $(5.2)_d$, $d = e$ and $\dim Y = i$.

Suppose $Z \neq X$ is irreducible and

$$Y \subset Z \subset X.$$

I claim that $Z = Y$. To see this, if we pass to an open affine subset then Z and Y are defined by ideals $J \subset I \subset A$, where A is the coordinate ring, $I = \langle f \rangle$ is principal and J is a prime ideal. Pick $g \in J$, $g \neq 0$. Write $g = g_1 g_2 \dots g_k$ as a product of irreducibles. As J is a prime ideal, $g_i \in J$ for some i. As $g_i \in I$, $g_i = uf$, and u must be a unit as g_i is irreducible. But then $I = J$ and $Z = Y$.

It follows that $i = d - 1$ and so $\dim Y = d - 1$. $\square$

Lemma 5.6. $(5.2)_{d-1}$ *implies* $(5.4)_d$.

Proof. The result is local on X, so we might as well assume that X and Y are irreducible and affine. We first show that

$$\mu(p) \geq \dim(X) - \dim(Y),$$

for every point of $p \in X$. If $e = \dim(Y) = \dim(X) = d$ there is nothing to prove. So we may assume that $e = \dim(Y) < d = \dim(X)$. Let $q = \pi(p)$. By (5.5) we may embed $Y \subset \mathbb{A}^n$ and pick a hyperplane $q \in H \subset Y$ such that $\dim(H \cap Y) = \dim(Y) - 1$. By an obvious induction, we may pick $\dim(Y)$ hyperplanes $H_1, H_2, \dots, H_e$, whose intersection is a finite set containing q. Working locally about q, we may assume that q is the only point in the intersection. Let $f_1, f_2, \dots, f_e$ be the corresponding polynomials. Then the fibre X_p is is defined by the polynomials $g_1, g_2, \dots, g_e$, where $g_i = \pi^* f_i$. So

$$\dim(X_p) \geq \dim(X) - \dim(Y),$$

as required.

To finish the proof, by Noetherian induction applied to X, it suffices to prove that there is an open subset U of X such that

$$\mu(p) \leq \dim(X) - \dim(Y),$$

for every $p \in U$. As usual, we may assume that $X \subset Y \times \mathbb{A}^n$ and that π is projection onto the second factor. Factoring π into the product of n projections, we may assume that $n = 1$, by induction on n. We may assume that $X \subset Y \times \mathbb{A}^1$ is closed. If $X = Y \times \mathbb{A}^1$ then $\mu_0 = 1$ and it is clear that $\dim X \geq \dim Y + 1$. As we have already proved the reverse inequality, $\dim X = \dim Y + 1$.

Otherwise there is a fibre of dimension zero. As X is a proper subset of Y, $\dim X = \dim Y$ and $\mu_0 = 0$. Working locally, we may assume that X is defined by polynomials of the form $F \in A(Y)[S, T]$. Further there is a polynomial $F \in A(Y)[S, T]$ vanishing on X, such that F_y is not the zero polynomial, for at least one $y \in Y$. In this case, the set of points where F_y is not the zero polynomial, is an open subset of Y, and for any point in this open subset, the fibre has dimension zero. $\square$

Lemma 5.7. $(5.4)_d$ *implies* $(5.2)_d$.

Proof. We may assume that X is affine. Pick a finite projection down to $\mathbb{A}^n$. As we are assuming $(5.4)_d$, $n = d$. It clearly suffices to prove the result for $X = \mathbb{A}^d$. Consider projection down to $\mathbb{A}^{d-1}$. Given a maximal chain of irreducible subsets

$$\varnothing \neq Z_0 \subset Z_1 \subset \dots \subset Z_n = \mathbb{A}^d,$$

let

$$\varnothing \neq Y_0 \subset Y_1 \subset \dots \subset Y_n = \mathbb{A}^{d-1},$$

be the image in $\mathbb{A}^{d-1}$. Then there is an index i such that Z_i contains the general fibre and Z_{i-1} does not contain the general fibre. Other than that, Y_j determines Z_j and the result follows by induction on d. $\square$

Proof of (5.2) *and* (5.4). Immediate from (5.6) and (5.7). $\square$

Corollary 5.8. *Let $\pi\colon X \longrightarrow Y$ be a surjective and projective morphism of quasi-projective varieties. Then the function*

$$\lambda\colon Y \longrightarrow \mathbb{N},$$

is upper semi-continuous, where $\lambda(p)$ is the dimension of the fibre $X_p = \pi^{-1}(X)$ at p. Moreover if X_0 is any irreducible component of X, with image Y_0, then we have

$$\dim(X_0) = \dim(Y_0) + \lambda_0,$$

where λ_0 is the minimum value of λ on Y_0.

Proof. π is proper as it is projective. Therefore the set

$$\{\, y \in Y \mid \lambda(y) \geq k \,\},$$

is closed as it is the image of the set

$$\{\, x \in X \mid \mu(x) \geq k \,\},$$

which is closed by (5.4). □

Note that we cannot discard the hypothesis that π is projective in (5.8). For example, let X be the disjoint union of $\mathbb{A}^2$ minus the y-axis and a single point p. Define a morphism $\pi\colon X \longrightarrow Y = \mathbb{A}^1$ by sending the extra point to the origin and otherwise taking the projection onto the x-axis. Then the fibre dimension is one at every point of Y, other than at the origin, where it is zero. In particular λ is not upper semi-continuous in this example. On the other hand, μ is upper semi-continuous, by virtue of the fact that the extra point is isolated in X.

One rather beautiful consequence of (5.4) is the following:

Corollary 5.9. *Let $\pi\colon X \longrightarrow Y$ be a morphism of projective varieties.*

If Y is irreducible and every fibre of π is irreducible and of the same dimension, then X is irreducible.

Proof. Let $X = X_1 \cup X_2 \cup \cdots \cup X_k$ be the decomposition of X into its irreducible components. Let $\pi_i = \pi|_{X_i}\colon X_i \longrightarrow Y_i$, where Y_i is the image of X_i and let $\lambda_i\colon X_i \longrightarrow \mathbb{N}$ be the function associated to π_i, as in (5.8). Let

$$Z_i = \{\, y \in Y_i \mid \lambda_i(y) \geq \lambda_0 \,\}.$$

(5.8) implies that the closed sets $Z_1, Z_2, \dots, Z_k$ cover Y. As Y is irreducible it follows that there is an index i, say $i = 1$, such that $Z_1 = Y_1 = Y$. But then the fibres of π_1 and π are equal, as they are of the same dimension and the fibres of π are irreducible. This is only possible if $X = X_1$. □

Example 5.10. *$\mathbb{P}^n$ has dimension n. More generally a toric variety containing a torus $\mathbb{G}_m^n$ has dimension n. In particular the toric variety corresponding to a fan F in N is equal to the rank of the free abelian group N.*

Consider $\mathbb{G}(k, n)$. Then this contains an open subset U isomorphic to $\mathbb{A}^{(k+1)(n-k)}$. So $\mathbb{G}(k, n)$ has dimension $(k+1)(n-k)$. For example, $\mathbb{G}(1, 3)$ has dimension $2 \cdot 2 = 4$.

Suppose that X and Y are quasi-projective varieties. Then the dimension of $X \times Y$ is the sum of the dimensions.

We can use (5.4) to calculate the dimension using different methods. One way is to project onto a linear subspace. If we repeatedly project from a point (which is the same as projecting once from a linear space of positive dimension) then the induced morphism $X \longrightarrow \mathbb{P}^k$ will eventually become dominant. At this point the morphism is finite over an open subset and the dimension of X is then k. Note that if we go back one step, then the closure of the image of X will be a hypersurface in $\mathbb{P}^{k+1}$.

Equivalently, if $X \subset \mathbb{P}^n$ and X has dimension d then a general linear space of dimension $n-d-1$ is disjoint from X and a general linear space of dimension $n-d$ meets X in a finite set of points. Note that general means that the linear space belongs to an open set of the corresponding Grasmannian. If X is closed, we can do slightly better, since if X is closed of dimension d, then every linear space of dimension $n-d$ must intersect X.

To calculate the dimension of an algebraic variety one can also use:

Definition 5.11. *Let L/K be a field extension. The* ***transcendence degree*** *of L/K is equal to the supremum of the length $x_1, x_2, \dots, x_k$ of algebraically independent elements of L/K.*

It is easy to prove:

Theorem 5.12. *Let X be an irreducible quasi-projective variety.*

Then the dimension of X is equal to the transcendence degree of $K(X)/K$.

One trick to calculate dimensions is to use the generic point of a variety. If we have a morphism $\pi\colon X \longrightarrow Y$ of irreducible varieties then μ_0 is actually the dimension of the generic fibre X_η, over the residue field of the generic point η of Y. Indeed the generic point ξ of X maps to the generic point of Y and so ξ is also the generic point of the generic fibre. The dimension of the generic fibre is the transcendence degree of the residue field of ξ over the residue field of η. The dimension of X

is the transcendence degree of the residue field of ξ over K. But the transcendence degree is additive on extensions.

Perhaps an easy example will make all of this clear. Consider $\mathbb{A}^2_K$. Suppose the generic point is ξ, with residue field $K(x, y)$. This has transcendence degree two over K. If we take a projection down to $\mathbb{A}^1_K$, with generic point η and residue field $K(y)$ then the transcendence degree of $K(x, y)/K(y)$ is one, the dimension of the generic fibre. $K(y)/K$ also has transcendence degree one and $\mathbb{A}^1_K$ has dimension one, as expected.

Now let's turn to calculating the dimension of some more examples, using these new techniques. Let us first calculate the dimension of the universal family over the Grassmannian.

$$\begin{tikzcd} \Sigma \arrow[r, "q"] \arrow[d, "p"'] & \mathbb{P}^n \\ \mathbb{G}(k, n). & \end{tikzcd}$$

Note that there are two ways to proceed; we can either use the morphism p or q.

First we use the morphism p. If we fix an element $[\Lambda] \in \mathbb{G}(k, n)$ then the fibre of p will be a copy of the k-plane Λ. Thus every fibre of p is isomorphic to $\mathbb{P}^k$. It follows that Σ has dimension $k + (k + 1)(n - k)$.

Now let us use the morphism q. If we fix point $x \in \mathbb{P}^n$, then the fibre of q is equal to the set of k-planes in $\mathbb{P}^n$, containing x. This is isomorphic to a Grassmannian $\mathbb{G}(k - 1, n - 1)$. Thus the dimension of Σ is equal to $n + k(n - k)$, which is easily seen to be equal to the previous expression.

Note that also we can prove that Σ is irreducible. Either way, it fibres over an irreducible base, with irreducible fibres of the same dimension.

Similarly the universal family of conics has dimension six (=five+one=two+four) and this space is irreducible. It is perhaps more interesting to figure out the dimension of the secant variety and the space of incident l-planes to an irreducible projective variety $X \subset \mathbb{P}^n$.

First the space $\mathcal{C}_l(X)$ of l-planes which meets a closed subset X of $\mathbb{P}^n$. In this case the universal family over $\mathcal{C}_l(X)$ has dimension equal to

$$\dim X + l(n - l),$$

where the second factor is equal to the dimension of the space of l-planes which contains a point. Since we have already seen that this is a variety isomorphic to $\mathbb{G}(l - 1, n)$, it follows that the universal family is irreducible, provided X is irreducible.

In particular suppose that X has dimension k, and suppose that $l \leq n-k-1$. Then a typical l-plane which meets X, will only meet X in one point. Thus the map from the universal family to $\mathbb{G}(l,n)$ is in fact birational, and the dimension of $\mathcal{C}_l(X)$ is

$$k + l(n-1).$$

In other words the codimension of $\mathcal{C}_l(X)$ is

$$n - l - k.$$

Thus if $l = n-k-1$, $\mathcal{C}_l(X)$ is a hypersurface in $\mathbb{G}(l,n)$.

Question 5.13. *Fix d. What is the smallest positive integer k such that any polynomial $f(x)$ of degree d over the field $\mathbb{C}$ is a sum of k dth powers of linear forms?*

One way to answer this problem is to use the secant variety to the rational normal curve of degree d. Let V be a two dimensional complex vector space. Then $\mathbb{P}^1 = \mathbb{P}(V)$ and the rational normal curve is the set of pure dth powers in the vector space $\mathbb{P}^d = \mathbb{P}(\mathrm{Sym}^d V)$ A polynomial $f(x)$ of degree d corresponds to a point of $\mathbb{P}^d$ and it is a sum of k dth powers if and only if belongs to the locus of $k-1$-planes which intersect C in k points. We want to know when this locus is the whole of $\mathbb{P}^n$. In this case its dimension is n.

It turns out that even when look at the locus of secant lines that this problem is very hard for a general variety X. In general, we have a rational morphism

$$X \times X \dashrightarrow \mathbb{G}(1,n)$$

Now note that if $l \subset \mathbb{G}(1,n)$ is a point of the image, then this map is not finite over l iff l is contained in X. Since the only subvariety with the property that the line through every two points is contained in the subvariety, is a linear space, we may assume that this map has finite fibres over an open set of the image. Then the image has dimension $2k$, where k is the dimension of X. Then the universal family over the image, has dimension $2k+1$ and the dimension of the image in $\mathbb{P}^n$ then has dimension $2k+1$ as well, provided that through a general point of the secant variety (the closure of the set of lines that meet X in at least two points), there passes only finitely secant lines.

Thus the expected dimension of the secant variety is $2k+1$, provided this dimension is at most n. For example, the secant variety to a space curve is expected to be the whole of $\mathbb{P}^3$ and the secant variety to a surface in $\mathbb{P}^5$ is expected to be the whole of $\mathbb{P}^5$.

Definition 5.14. *Let X be a closed irreducible non-degenerate (that is X is not contained in a proper linear subspace) subvariety of $\mathbb{P}^n$.*

The ***deficiency of*** X*, denoted* $\delta(X)$*, is equal to the dimension of the family of secant lines passing through a general point of the secant variety.*

We have already seen then that the dimension of the secant variety is equal to $2k + 1 - \delta(X)$.

Let us calculate the secant variety to the d-uple embedding, at least in characteristic zero. Recall the if $X = \mathbb{P}(V) = \mathbb{P}^k$ then X is embedded in $\mathbb{P}(\mathrm{Sym}^d(V))$, as the space of rank one symmetric tensors (the pure powers). The secant variety then consists of all rank at most two symmetric tensors, that is anything which is a sum of two rank one symmetric tensors.

In the case of the Veronese, we get the space of rank two quadratic forms. As there are quadratic forms of rank three, it follows than the secant variety to the Veronese is a proper subset of $\mathbb{P}^5$. In fact the space of rank two symmetric tensors is a hypersurface in $\mathbb{P}^5$, given as the vanishing of a determinant. Expanding it follows that the secant variety is defined by a cubic polynomial. Note that the deficiency is equal to 1 in this case.

It is interesting to look at the dimension of some more exotic schemes. $\operatorname{Spec}\mathbb{Z}$ has dimension. Consider $\mathbb{A}^1_{\mathbb{Z}}$. This has dimension one over $\operatorname{Spec}\mathbb{Z}$ and absolute dimension two. Consider $\mathbb{A}^2_{\mathbb{Z}}$. This has absolute dimension two over $\operatorname{Spec}\mathbb{Z}$ and so it has absolute dimension three.