4. Coherent Sheaves

Definition 4.1. *If $(X, \mathcal{O}_X)$ is a locally ringed space, then we say that an $\mathcal{O}_X$-module $\mathcal{F}$ is* ***locally free*** *if there is an open affine cover $\{U_i\}$ of X such that $\mathcal{F}|_{U_i}$ is isomorphic to a direct sum of copies of $\mathcal{O}_{U_i}$. If the number of copies r is finite and constant, then $\mathcal{F}$ is called* ***locally free of rank*** *r (aka a* ***vector bundle****).*

If $\mathcal{F}$ is locally free of rank one then we way say that $\mathcal{F}$ is ***invertible*** *(aka* ***a line bundle****). The group of all invertible sheaves under tensor product, denoted $\operatorname{Pic}(X)$, is called the* ***Picard group*** *of X.*

A ***sheaf of ideals*** *$\mathcal{I}$ is any $\mathcal{O}_X$-submodule of $\mathcal{O}_X$.*

Definition 4.2. *Let $X = \operatorname{Spec} A$ be an affine scheme and let M be an A-module. $\tilde{M}$ is the sheaf which assigns to every open subset $U \subset X$, the set of functions*

$$s\colon U \longrightarrow \coprod_{\mathfrak{p}\in U} M_{\mathfrak{p}},$$

which can be locally represented at $\mathfrak{p}$ as a/g, $a \in M$, $g \in R$, $\mathfrak{p} \notin U_g \subset U$.

Lemma 4.3. *Let A be a ring and let M be an A-module. Let $X = \operatorname{Spec} A$.*

1. *$\tilde{M}$ is a $\mathcal{O}_X$-module.*
2. *If $\mathfrak{p} \in X$ then $\tilde{M}_{\mathfrak{p}}$ is isomorphic to $M_{\mathfrak{p}}$.*
3. *If $f \in A$ then $\tilde{M}(U_f)$ is isomorphic to M_f.*

Proof. (1) is clear and the rest is proved mutatis mutandis as for the structure sheaf. □

Definition 4.4. *An $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is called* ***quasi-coherent*** *if there is an open cover $\{U_i = \operatorname{Spec} A_i\}$ by affines and isomorphisms $\mathcal{F}|_{U_i} \simeq \tilde{M}_i$, where M_i is an A_i-module. If in addition M_i is a finitely generated A_i-module then we say that $\mathcal{F}$ is* ***coherent****.*

Proposition 4.5. *Let X be a scheme. Then an $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if for every open affine $U = \operatorname{Spec} A \subset X$, $\mathcal{F}|_U = \tilde{M}$. If in addition X is Noetherian then $\mathcal{F}$ is coherent if and only if M is a finitely generated A-module.*

This is proved using almost the same techniques as the proof for the structure sheaf; the key point is that if a collection of sections of $\mathcal{O}_X$ don't vanish simultaneously then we can write 1 as a linear combination of these sections.

Theorem 4.6. *Let $X = \operatorname{Spec} A$ be an affine scheme.*

The assignment $M \longrightarrow \tilde{M}$ defines an equivalence of categories between the category of A-modules to the category of quasi-coherent sheaves on X, which respects exact sequences, direct sum and tensor product, and which is functorial with respect to morphisms of affine schemes, $f\colon X = \operatorname{Spec} A \longrightarrow Y = \operatorname{Spec} B$. If in addition A is Noetherian, this functor restricts to an equivalence of categories between the category of finitely generated A-modules to the category of coherent sheaves on X.

Theorem 4.7. *Let X be a scheme.*

The kernel and cokernel of a morphism between two quasi-coherent sheaves is quasi-coherent. An extension of quasi-coherent sheaves is quasi-coherent, that is, if the two outer terms of a short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0,$$

are quasi-coherent then so is middle.

If X is Noetherian then one can replace quasi-coherent by coherent.

Proof. Since this result is local, we may assume that $X = \operatorname{Spec} A$ is affine. The only non-trivial thing is to show that if $\mathcal{F}$ and $\mathcal{H}$ are quasi-coherent then so is $\mathcal{G}$. By (II.5.6) of Hartshorne, there is an exact sequence on global sections,

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0.$$

It follows that there is a commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & \tilde{F} \arrow[r] \arrow[d] & \tilde{G} \arrow[r] \arrow[d] & \tilde{H} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{H} \arrow[r] & 0,
\end{tikzcd}$$

whose rows are exact. By assumption, the first and third vertical arrow are isomorphisms, and the 5-lemma implies that the middle arrow is an isomorphism. $\square$

Lemma 4.8. *Let $f\colon X \longrightarrow Y$ be a scheme.*

1. *If $\mathcal{G}$ is a quasi-coherent sheaf (respectively X and Y are Noetherian and $\mathcal{G}$ is coherent) on Y then $f^*\mathcal{G}$ is quasi-coherent (respectively coherent).*
2. *If $\mathcal{F}$ is a quasi-coherent sheaf on X and either f is compact and separated or X is Noetherian then $f_*\mathcal{F}$ is quasi-coherent.*

Proof. (1) is local on both X and Y and so follows easily from the affine case.

(2) is local on Y, so we may assume that Y is affine. By assumption (either way) X is compact and so we may cover X by finitely many

open affines $\{U_i\}$. If f is separated then $U_i \cap U_j$ is affine. Otherwise X is Noetherian and we can cover $U_i \cap U_j$ by finitely many open affines U_{ijk}. If $V \subset Y$ is open then a section s of $\mathcal{F}$ on the open set $f^{-1}(V)$ is the same as to give sections of $\mathcal{F}$ on the open cover $\{f^{-1}(V) \cap U_i\}$ which agree on overlaps $\{f^{-1}(V) \cap U_{ijk}\}$ (this is simply a restatement of the sheaf axiom). It follows that there is an exact sequence of sheaves

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \bigoplus_i f_*(\mathcal{F}|_{U_i}) \longrightarrow \bigoplus_{ijk} f_*(\mathcal{F}|_{U_{ijk}}).$$

The last two sheaves are quasi-coherent, since U_i and U_{ijk} are coherent and a direct sum of quasi-coherent sheaves is quasi-coherent. But then the first term is quasi-coherent, by (4.7). □

Definition-Lemma 4.9. *Let X be a scheme. If $Y \subset X$ is a closed subscheme, then the kernel of the morphism of sheaves*

$$\mathcal{O}_X \longrightarrow \mathcal{O}_Y,$$

*defines a quasi-coherent ideal sheaf $\mathcal{I}_Y$, called the **ideal sheaf of** Y **in** X, which is coherent if X is Noetherian.*

Conversely if $\mathcal{I} \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals then there is a closed subscheme Y of X such that $\mathcal{I}$ is the ideal sheaf of Y in X.

Proof. If $Y \subset X$ is a closed subscheme then $\mathcal{I}_Y$ is a quasi-coherent sheaf, by (4.7), which is coherent if X is Noetherian.

Now suppose that $\mathcal{I}$ is quasi-coherent. Let Y be the support of the quotient sheaf $\mathcal{O}_X/\mathcal{I}$. Then Y is a closed subset. Uniqueness is clear. We have to check that $(Y, \mathcal{O}_X/\mathcal{F})$ is a closed subscheme. We may check this locally so that we may assume that $X = \operatorname{Spec} A$ is affine. Then $\mathcal{I} = \tilde{I}$, where $I = \Gamma(X, \mathcal{I}) \subset A$ is an A-submodule, that is an ideal. $(Y, \mathcal{O}_X/\mathcal{I})$ is then the closed affine subscheme corresponding to I. □

Remark 4.10. *Let $i\colon Y \longrightarrow X$ be a closed subscheme. If $\mathcal{F}$ is a sheaf on Y, then $\mathcal{G} = i_*\mathcal{F}$ is a sheaf on X, whose support is contained in Y. Conversely, given any sheaf $\mathcal{G}$ on X, whose support is contained in Y, then there is a unique sheaf $\mathcal{F}$ on Y such that $i_*\mathcal{F} = \mathcal{G}$.*

For this reason, it is customary, as in (4.9), to abuse notation, and to not distinguish between sheaves on Y and sheaves on X, whose support is contained in Y.

Note also that (4.9) implies that the closed subschemes $Y \subset X =$ $\operatorname{Spec} A$ of an affine scheme are in bijection with the ideals $I \trianglelefteq A$.

Definition 4.11. *Let $f\colon X \longrightarrow S$ be a morphism of schemes. We say that f is **affine** if there is an open affine cover $\{S_i\}$ of S such that $f^{-1}(S_i)$ is an affine open subset of X.*

Remark 4.12. *It is straightforward to show that f is affine if and only if for every open affine subset $V \subset S$, $f^{-1}(V)$ is affine. Note that if $f\colon X \longrightarrow S$ is affine then $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras.*

Let S be a scheme and let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Take an open affine cover $\{S_i = \operatorname{Spec} R_i\}$ of S. As $\mathcal{A}$ is quasi-coherent $\mathcal{A}|_{S_i} \simeq \tilde{A}_i$, for some R_i-algebra A_i. This gives a morphism or affine schemes $f_i\colon X_i = \operatorname{Spec} A_i \longrightarrow S_i$. By composition this gives a morphism $X_i \longrightarrow S$. It is straightforward to check that we can glue these morphisms together to get a scheme $X = \mathbf{Spec}\,\mathcal{A}$ and an affine morphism $f\colon X \longrightarrow S$.

Theorem 4.13. *Fix a scheme S. There is an equivalence of categories between affine morphisms $f\colon X \longrightarrow S$ and quasi-coherent sheaves of $\mathcal{A} = \mathcal{O}_S$-algebras.*

Let $\mathcal{Q}$ be a locally free sheaf of rank r on a scheme S. We can construct the symmetric algebra $\operatorname{Sym} \check{\mathcal{Q}}$. This is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Let $X = \mathbf{Spec}(\operatorname{Sym} \check{\mathcal{Q}})$. The fibres of the affine morphism $f\colon X \longrightarrow S$ are affine spaces of dimension r. In fact, if $\mathcal{Q}$ is the trivial sheaf of rank r then $X = \mathbb{A}^r_S$, so that if $\{S_i\}$ is an open affine cover of S such that $\mathcal{Q}_i$ is the trivial sheaf of rank r then $X_i = \mathbb{A}^r_{S_i}$. Intuitively f is a fibre bundle, with fibres isomorphic to affine space. In fact f comes with a distinguished section and in fact X is (what is known as) a vector bundle of rank r over S. All of this discussion motivates the following:

Definition 4.14. *Let B a scheme. A **family of k-planes over B in an n-dimensional vector space** is a morphism of locally free sheaves*

$$\bigoplus_{i=1}^{n} \mathcal{O}_B \longrightarrow \mathcal{Q},$$

where $\mathcal{Q}$ has rank k.

Note that if we take global spec then we get a map of vector bundles, from the trivial vector bundle of rank n to a vector bundle of rank k. Note also that we need to work with quotient sheaves. The problem is that if we take the vector bundles associated to an inclusion of locally free sheaves this need not gives a map of vector bundles (recall that $V \subset E$ is a sub-vector bundle if the quotient vector bundle exists).

Definition 4.15. *Fix a scheme S. Let F be the functor from the category of schemes over S to (Sets) which assigns to every scheme B*

over S the set of all isomorphism classes of families of k-planes over B in an n-dimensional space.

Theorem 4.16. *There is a integral projective scheme $G_S(k,n)$ which represents the functor F.*

Note that the Grassmanian comes equipped with a locally free sheaf $\mathcal{Q}$ of rank k, which is a quotient of the trivial locally free sheaf of rank n. This sheaf is called the **universal quotient sheaf**.

Note also that the definition of the Grassmanian is inconsistent with the classical definition of projective space over an algebraically closed field K. If one follows the definition given above, the closed points of $\mathbb{P}^n_K$ with residue field K are surely the one dimensional quotients of K^{n+1} and not the one dimensional subspaces. If one adopts the functorial approach of schemes we have no choice but to define everything in terms of quotient spaces. For example, suppose we consider $\mathbb{P}^1_{\mathbb{Z}}$. Then we want to look at one dimensional objects attached to $\mathbb{Z}^2$. If we try to work with subgroups of rank one then we run into trouble,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^2 \longrightarrow Q \longrightarrow 0.$$

If the inclusion map is given by $1 \longrightarrow (1,0)$ then $Q \simeq \mathbb{Z}$ as expected. But if we consider something like $1 \longrightarrow (2,0)$ then $Q \simeq \mathbb{Z}_2 \oplus \mathbb{Z}$, which is not correct. However if we consider sequences of the form

$$0 \longrightarrow K \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

then K is always isomorphic to $\mathbb{Z}$. On the other hand, it would seem impractical to change the classical definition of projective space in anticipation of this problem.

Most of what we have done with algebras and modules, makes sense for graded algebras and graded modules, in which case we get sheaves on proj of the graded ring.

Definition 4.17. *Let S be a graded ring and let M be a graded S-module. If $\mathfrak{p} \triangleleft S$ is a homogeneous ideal, then $M_{(\mathfrak{p})}$ denotes those elements of the localisation $M_{\mathfrak{p}}$ of degree zero.*

$\tilde{M}$ is the sheaf on $\operatorname{Proj} S$, which given an open subset $U \subset \operatorname{Proj} S$, assigns the set $\tilde{M}(U)$ of those functions

$$s\colon U \longrightarrow \coprod_{\mathfrak{p} \in U} M_{(\mathfrak{p})},$$

which are locally fractions of degree zero.

Proposition 4.18. *Let S be a graded ring, let M be a graded S-module and let $X = \operatorname{Proj} S$.*

(1) *For any* $\mathfrak{p} \in X$, $(\tilde{M})_{\mathfrak{p}} \simeq M_{(\mathfrak{p})}$.
(2) *If* $f \in S$ *is homogeneous,*
$$(\tilde{M})_{U_f} \simeq \tilde{M}_{(f)}.$$
(3) $\tilde{M}$ *is a quasi-coherent sheaf. If* S *is Noetherian and* M *is finitely generated then* $\tilde{M}$ *is a coherent sheaf.*

Definition 4.19. *Let* $X = \operatorname{Proj} S$, *where* S *is a graded ring. If* n *is any integer, then set*
$$\mathcal{O}_X(n) = S(n)\tilde{}.$$
If $\mathcal{F}$ *is any sheaf of* $\mathcal{O}_X$*-modules,*
$$\mathcal{F}(n) = \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(n).$$
Let
$$\Gamma_*(X, \mathcal{F}) = \bigoplus_{m \in \mathbb{N}} \Gamma(X, \mathcal{F}(n)).$$

Lemma 4.20. *Let* S *be a graded ring,* $X = \operatorname{Proj} S$ *and let* M *be a graded* S*-module.*

(1) $\mathcal{O}_X(n)$ *is an invertible sheaf.*
(2) $\tilde{M}(n) \simeq M(n)\tilde{}$. *In particular* $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \simeq \mathcal{O}_X(m+n)$.
(3) *Formation of the twisting sheaf* $\mathcal{O}_X(1)$ *is functorial with respect to morphisms of graded rings.*

Proposition 4.21. *Let* A *be a ring, let* $S = A[x_0, x_1, \dots, x_r]$ *and let* $X = \mathbb{P}^r_A = \operatorname{Proj} A[x_0, x_1, \dots, x_r]$.

Then
$$\Gamma_*(X, \mathcal{O}_X) \simeq S.$$

Lemma 4.22. *Let* S *be a graded ring, generated as an* S_0*-algebra by* S_1.

If $X = \operatorname{Proj} S$ *and* $\mathcal{F}$ *is a quasi-coherent sheaf on* X, *then*
$$\Gamma_*(X, \mathcal{F})\tilde{} = \mathcal{F}.$$

Theorem 4.23. *Let* A *be a ring.*

(1) *If* $Y \subset \mathbb{P}^n_A$ *is a closed subscheme then* $Y = \operatorname{Proj} S/I$, *for some homogeneous ideal* $I \subset S = A[x_1, x_2, \dots, x_n]$.
(2) Y *is projective over* $\operatorname{Spec} A$ *if and only if it is isomorphic to* $\operatorname{Proj} T$ *for some graded ring* T, *for which there are finitely many elements of* T_1 *which generate* T *as a* $T_0 = A$*-algebra.*

Proof. Let $\mathcal{I}_Y$ the ideal sheaf of Y in X. Then there is an exact sequence,

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Twisting by $\mathcal{O}_X(n)$ is exact (in fact $\mathcal{O}_X(n)$ is an invertible sheaf), so we get an exact sequence

$$0 \longrightarrow \mathcal{I}_Y(n) \longrightarrow \mathcal{O}_X(n) \longrightarrow \mathcal{O}_Y(n) \longrightarrow 0.$$

Taking global sections is left exact, so we get an exact sequence

$$0 \longrightarrow \mathcal{I}_Y(n) \longrightarrow \mathcal{O}_X(n) \longrightarrow \mathcal{O}_Y(n).$$

Taking the direct sum, there is therefore an injective map

$$I = \Gamma_*(X, \mathcal{I}_Y) = \Gamma_*(X, \mathcal{O}_X) \simeq S.$$

It follows that $I \triangleleft S$ is a homogeneous ideal. Let $\tilde{I}$ be the associated sheaf. Since $\mathcal{I}_Y$ is quasi-coherent, (4.22) implies that $\tilde{I} = \mathcal{I}_Y$. But then the subscheme determined by I is equal to Y. Hence (1).

If Y is projective over $\operatorname{Spec} A$ then we may assume that $Y \subset \mathbb{P}^n_A$. By (1) $Y \simeq \operatorname{Proj} S/I$, and if $T = S/I$, then $T_0 \simeq A$ and the images of $x_0, x_1, \dots, x_n \in T_1$ generate T. Conversely, any such algebra is the quotient of S. The kernel I is a homogeneous ideal and $Y \simeq \operatorname{Proj} S/I$. $\square$

Definition 4.24. *Let Y be a scheme. $\mathcal{O}_{\mathbb{P}^r_Y}(1) = g^*\mathcal{O}_{\mathbb{P}^r}(1)$ is the sheaf on $\mathbb{P}^r_Y$, where $g\colon \mathbb{P}^r_Y \longrightarrow \mathbb{P}^r_{\operatorname{Spec}\mathbb{Z}}$ is the natural morphism.*

We say that a morphism $i\colon X \longrightarrow Z$ is an ***immersion*** *if i induces an isomorphism of X with a locally closed subset of Y.*

We say that an invertible sheaf $\mathcal{L}$ on a scheme X over Y is ***very ample*** *if there is an immersion $i\colon X \longrightarrow \mathbb{P}^r_Y$ over Y, such that $\mathcal{L} \simeq i^*\mathcal{O}_{\mathbb{P}^r_Y}(1)$.*

Lemma 4.25. *Let X be a scheme over Y.*

Then X is projective over Y if and only if X is proper over Y and there is a very ample sheaf on X.

Proof. One direction is clear; if X is projective over Y, then it is proper and we just pullback $\mathcal{O}_{\mathbb{P}^r_Y}(1)$.

If X is proper over Y then the image of X in $\mathbb{P}^r_Y$ is closed, and so X is projective over Y. $\square$

Definition 4.26. *Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is* ***globally generated*** *if there are elements $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ such that for every point $x \in X$, the images of s_i in the stalk $\mathcal{F}_x$, generate the stalk as an $\mathcal{O}_{X,x}$-module.*

Lemma 4.27. *Let X be a scheme and*
TFAE

(1) *$\mathcal{F}$ is globally generated.*
(2) *The natural map*
$$H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \longrightarrow \mathcal{F},$$
is surjective.
(3) *$\mathcal{F}$ is a quotient of a free sheaf.*

Proof. Clear. □

Lemma 4.28 (Push-pull)**.** *Let $f\colon X \longrightarrow Y$ be a morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{G}$ be a locally free $\mathcal{O}_Y$-module.*
$$f_*(\mathcal{F} \underset{\mathcal{O}_X}{\otimes} f^*\mathcal{G}) = f_*\mathcal{F} \underset{\mathcal{O}_Y}{\otimes} \mathcal{G}.$$

Theorem 4.29 (Serre)**.** *Let X be a projective scheme over a Noetherian ring A, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

Then there is a positive integer $n_0 \geq 0$ such that $\mathcal{F}(n)$ is globally generated for all $n \geq n_0$.

Proof. By assumption there is a closed immersion $i\colon X \longrightarrow \mathbb{P}^r_A$ such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r_A}(1)$. Let $\mathcal{G} = i_*\mathcal{F}$. Then (4.28) implies that
$$\mathcal{G}(n) = i_*\mathcal{F}(n).$$
Then $\mathcal{F}(n)$ is globally generated if and only if $\mathcal{G}(n)$ is globally generated. As i is a closed immersion it is a proper morphism; as $\mathcal{F}$ is coherent, i is proper, and X and $\mathbb{P}^r_A$ are Noetherian, $\mathcal{G}$ is coherent. Replacing X by $\mathbb{P}^r_A$ and $\mathcal{F}$ by $\mathcal{G}$, we may assume that $X = \mathbb{P}^r_A$.

Consider the standard open affine cover U_i, $0 \leq i \leq r$ of $\mathbb{P}^r_A$. Since $\mathcal{F}$ is coherent, $\mathcal{F}_i = \mathcal{F}|_{U_i} = \tilde{F}_i$, for some finitely generated $A[X_0/X_i, X_1/X_i, \dots, X_r/X_i]$-module F_i. Pick generators s_{ij} of F_i. For each j, we may lift $X_i^{n_{ij}} s_{ij}$ to t_{ij}, for some n_{ij} (see (II.5.14)). By finiteness, we may assume that $n = n_{ij}$ does not depend on i and j. Now the natural map
$$X_i^n\colon \mathcal{F} \longrightarrow \mathcal{F}(n),$$
is an isomorphism over U_i. Thus t_{ij} generate the stalks of $\mathcal{F}$. □

Corollary 4.30. *Let X be a scheme projective over a Noetherian ring A and let $\mathcal{F}$ be a coherent sheaf.*

Then $\mathcal{F}$ is a quotient of a direct sum of line bundles of the form $\mathcal{O}_X(n_i)$.

Proof. Pick $n > 0$ such that $\mathcal{F}(n)$ is globally generated. Then

$$\bigoplus_{i=1}^{k} \mathcal{O}_X \longrightarrow \mathcal{F}(n),$$

is surjective. Now just tensor by $\mathcal{O}_X(-n)$. $\square$