3. Grasmannians

We first treat Grassmanians classically. Fix an algebraically closed field K. We want to parametrise the space of k-planes W in a vector space V. The obvious way to parametrise k-planes is to pick a basis $v_1, v_2, \dots, v_k$ for W. Unfortunately this does not specify W uniquely, as the same vector space has many different bases. However, the line spanned by the vector

$$\omega = v_1 \wedge v_2 \wedge \dots \wedge v_k \in \bigwedge^k V,$$

is invariant under re-choosing a basis.

Definition 3.1. *The **Grassmannian** $G(k, V)$ **of k-planes in** V is the set of rank one vectors in $\mathbb{P}(\bigwedge^k V)$.*

We set $G(k, n) = G(k, K^n)$ and $\mathbb{G}(k, n) = G(k+1, n+1)$. The latter may be thought of as the set of k-planes in $\mathbb{P}^n$.

Lemma 3.2. *The Grassmannian is a closed subset of $\mathbb{P}^N$.*

Proof. Consider the rational map

$$\prod^k \mathbb{P}(V) \dashrightarrow \mathbb{P}(\bigwedge^k V),$$

which sends $([v_1], [v_2], \dots [v_k])$ to $[v_1 \wedge v_2 \wedge \dots \wedge v_k]$. The image (that is, take the image of the graph) is the Grassmannian and the image under a rational map is closed. $\square$

The embedding of the Grassmannian inside $\mathbb{P}(\bigwedge^k V)$ is known as the Plücker embedding. If we choose a basis $e_1, e_2, \dots, e_n$ for V, then a general element of $\bigwedge^k V$ is given by

$$\sum_I p_I e_I,$$

where I ranges over all collections of increasing sequences of integers between 1 and n,

$$i_1 < i_2 < \dots < i_k,$$

and e_I is shorthand for the wedge of the corresponding vectors,

$$e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}.$$

The coefficients p_I are naturally coordinates on $\mathbb{P}(\bigwedge^k V)$, which are known as the Plücker coordinates.

There is another way to look at the construction of the Grassmannian which is very instructive. If we pick a basis $e_1, e_2, \dots, e_n$ for V, then let A be the $k \times n$ matrix whose rows are $v_1, v_2, \dots, v_k$, in this basis.

As before, this matrix does not uniquely specify $W \subset V$, since we could pick a new basis for W. However the operation of picking a new basis corresponds to taking linear combinations of the rows of our matrix, which in turn is the same as multiplying our matrix by a $k \times k$ invertible matrix on the left. In other words the Grassmannian is the set of equivalence classes of $k \times n$ matrices under the action of $\mathrm{GL}_k(K)$ by multiplication on the left.

It is not hard to connect the two constructions. Given the matrix A, then form all possible $k \times k$ determinants. Any such determinant is determined by specifying the columns to pick, which we indicate by a multindex I. In terms of $\bigwedge^k V$, this is the same as picking a basis and expanding our vector as a sum

$$\sum_I p_I e_I,$$

where, as before, e_I is the wedge of the corresponding vectors. For example consider the case $k = 2$, $n = 4$ (lines in $\mathbb{P}^3$). We have a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

The corresponding plane is given as the span of the rows. We can form six two by two determinants. Clearly these are invariant, up to scalars, under the action of $\mathrm{GL}_2(K)$.

The Grassmannian has a natural cover by open affine subsets, isomorphic to affine space, in much the same way that projective space has a cover by open affines, isomorphic to affine space. Pick a linear space U of dimension $n - k$, and consider the set of linear spaces W of dimension k which are complementary to U, that is, which meet U only at the origin. Identify V with the sum $V/U + U$. Then a linear space W complementary to U can be identified with the graph of a linear map

$$V/U \longrightarrow U.$$

It follows that the subset of all linear spaces W complementary to U is equal to

$$\mathrm{Hom}(V/U, U) \simeq K^{k(n-k)} \simeq \mathbb{A}_K^{k(n-k)}.$$

Another way to see this is as follows. Consider the first $k \times k$ minor. Suppose that the corresponding determinant is non-zero, that is the corresponding vectors are independent. In this case the $k \times k$ minor is equivalent to the identity matrix, and the only element of $\mathrm{GL}_k(K)$ which fixes the identity, is the identity itself. Thus we have a canonical representative of the matrix A for the linear space W. We are free to choose the other $k \times (n - k)$ block of the matrix, which gives us an affine space of dimension $k(n-k)$. The condition that the first $k \times k$ minor has non-zero determinant is an open condition, and this gives us an open affine cover by affine spaces of dimension $k(n-k)$. Note that the condition that the first $k \times k$ minor is invertible is equivalent to the condition that we do not meet the space given by the vanishing of the first k coordinates, which is indeed a linear space of dimension $n-k$.

It is interesting to write down the equations cutting out the image of the Grassmannian under the Plücker embedding, although this turns out to involve some non-trivial multilinear algebra. The problem is to characterise the set of rank one vectors ω in $\bigwedge^k V$.

Definition 3.3. *Let $\omega \in \bigwedge^k V$. We say that ω is* ***divisible*** *by $v \in V$ if there is an element $\phi \in \bigwedge^k V$ such that $\omega = \phi \wedge v$.*

Lemma 3.4. *Let $\omega \in \bigwedge^k V$.*

Then ω is divisible by v iff $\omega \wedge v = 0$.

Proof. This is easy. If $\omega = \phi \wedge v$, then

$$\begin{aligned} \omega \wedge v &= \phi \wedge v \wedge v \\ &= 0. \end{aligned}$$

To see the other direction, extend v to a basis $v_1 = e_1, e_2, \dots, e_n$ of V. Then we may expand ω in this basis.

$$\omega = \sum p_I e_I.$$

On the other hand

$$e_I \wedge v = \begin{cases} e_J & \text{if } 1 \notin I \text{, where } J = \{1\} \cup I \\ 0 & \text{if } 1 \in I. \end{cases}$$

Thus $\omega \wedge v = 0$ iff $p_I \neq 0$ implies $1 \in J$ iff v divides ω. ☐

Lemma 3.5. *Let $\omega \in \bigwedge^k V$.*

Then ω has rank one iff the linear map

$$\phi(\omega) \colon V \longrightarrow \bigwedge^{k+1} V \qquad\qquad v \longrightarrow \omega \wedge v,$$

has rank at most $n-k$.

Proof. Indeed $\phi(\omega)$ has rank at most $n-k$ iff the linear subspace of vectors dividing ω has dimension at least k iff ω has rank one. ☐

Now the map

$$\phi \colon \bigwedge^{k} V \longrightarrow \operatorname{Hom}(V, \bigwedge^{k+1} V),$$

is clearly linear. Thus the map ϕ can be interpreted as a matrix whose entries are linear coordinates of $\bigwedge^k V$ and the locus we want is given by the vanishing of the $(n-k+1) \times (n-k+1)$ minors.

Unfortunately the equations we get in this way won't be best possible. In particular they won't generate the ideal of the Grassmannian (they only cut out the Grassmannian set theoretically). To find equations that generate the ideal, we have to work quite a bit harder.

Lemma 3.6. *There is a natural pairing between $\bigwedge^k V$ and $\bigwedge^{n-k} V^*$. This pairing is well-defined up to scalars and preserves the rank.*

Proof. There is a natural pairing

$$\bigwedge^{k} V \times \bigwedge^{n-k} V \longrightarrow \bigwedge^{n} V,$$

which sends

$$(\omega, \eta) \longrightarrow \omega \wedge \eta.$$

On the other hand, $\bigwedge^n V$ is one dimensional so that it is non-canonically isomorphic to K and $(\bigwedge^{n-k} V)^*$ is isomorphic to $\bigwedge^{n-k} V^*$. $\square$

Given ω, let ω^* be the corresponding element of $\bigwedge^{n-k} V^*$. Now there is a natural map

$$\psi(\omega^*)\colon V^* \longrightarrow \bigwedge^{n-k+1} V^*$$

which sends

$$v^* \longrightarrow \omega^* \wedge v^*.$$

Further ω has rank one iff ω^* has rank one, which occurs if and only if $\psi(\omega^*)$ has rank k.

Moreover the kernel of $\phi(\omega)$, namely W, is precisely the annihilator of the kernel of $\psi(\omega^*)$. Dualising, we get maps

$$\phi^*(\omega)\colon \bigwedge^{k+1} V^* \longrightarrow V^* \qquad \text{and} \qquad \psi^*(\omega)\colon \bigwedge^{n-k+1} V \longrightarrow V,$$

whose images annihilate each other.

Thus ω has rank one iff for every $\alpha \in \bigwedge^{k+1} V^*$ and $\beta \in \bigwedge^{n-k+1} V^*$,

$$\Xi_{\alpha,\beta}(\omega) = \langle \phi^*(\omega)(\alpha), \psi^*(\omega)(\beta) \rangle = 0.$$

Now $\Xi_{\alpha,\beta}$ are quadratic polynomials, which are known as the Plücker relations. It turns out that they do indeed generate the ideal of the Grassmannian.

It is interesting to see what happens when $k = 2$:

Lemma 3.7. *Let $\omega \in \bigwedge^2 V$.*

Then ω has rank one iff $\omega \wedge \omega = 0$.

Proof. One direction is clear, in fact for every k, if ω has rank one then $\omega \wedge \omega = 0$.

To see the other direction, we need to prove that if ω has rank at least two, then $\omega \wedge \omega \neq 0$. First observe that if ω has rank at least two, then we may find a projection down to a vector space of dimension four, such that the image has rank two. Thus we may assume that V has dimension four and ω has rank two. In this case, up to change of coordinates,

$$\omega = e_1 \wedge e_2 + e_3 \wedge e_4,$$

and by direct computation, $\omega \wedge \omega$ is not zero. $\square$

Now

$$\omega = \sum_{i,j} p_{i,j} e_i \wedge e_j.$$

Suppose that $n = 4$. If we expand

$$\omega \wedge \omega,$$

then everything is a multiple of $e_1 \wedge e_2 \wedge e_3 \wedge e_4$. We need to pick a term from each bracket, so that the union is $\{1, 2, 3, 4\}$. In other words, the coefficient of the expansion is a sum over all partitions of $\{1, 2, 3, 4\}$ into two equal parts. By direct computation, we get

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}.$$

In particular, $\mathbb{G}(1, 3)$ is a quadric in $\mathbb{P}^5$, of maximal rank. Unfortunately this also makes it clear that the Grassmannian is not a toric variety (if it were, it would be defined by a binomial, not a trinomial). It turns out that the Grassmannian is close to a toric variety (it is a spherical variety). In fact the algebraic group $\mathrm{GL}_n(V)$ acts transitively on $G(k, V)$. The stabiliser subgroup H of the k-plane $W \subset V$ spanned by the first k vectors is given by invertible matrices of the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

So

$$G(k, V) = \mathrm{GL}_n(V)/H.$$

As with the space of conics in $\mathbb{P}^2$, the main point of the Grassmannian, is that it comes with a universal family. We first investigate what this means in the baby case of quasi-projective varieties before we move on to the more interesting case of schemes.

Definition 3.8. *A* ***family of k-planes in $\mathbb{P}^n$ over*** *B is a closed subset $\Sigma \subset B \times \mathbb{P}^n$ such that the fibres, under projection to the first factor, are identified with k-planes in $\mathbb{P}^n$.*

Definition 3.9. *Let F be the functor from the category of varieties to the category of sets, which assigns to every variety, the set of all (flat) families of k-planes in $\mathbb{P}^n$, up to isomorphism.*

Theorem 3.10. *The Grassmannian $\mathbb{G}(k,n)$ represents the functor F.*

It might help to unravel some of the definitions. Suppose that we are given a variety B. Essentially we have to show that there is a natural bijection of sets,

$$F(B) = \operatorname{Hom}(B, \mathbb{G}(k,n)).$$

The set on the left is nothing more than the set of all families of k-planes in $\mathbb{P}^n$, with base B. In particular given a morphism $f\colon B \longrightarrow \mathbb{G}(k,n)$, we are supposed to produce a family of k-planes over B. Here is how we do this. Suppose that we have constructed the natural family of k-planes over $\mathbb{G}(k,n)$,

$$\begin{tikzcd} \Sigma \arrow[r, hook] & \mathbb{G}(k,n) \times \mathbb{P}^n \arrow[d] \\ & \mathbb{G}(k,n), \end{tikzcd}$$

so that the fibre over $[\Lambda] \in \mathbb{G}(k,n)$ is exactly the set,

$$\{[\Lambda]\} \times \Lambda \subset \{[\Lambda]\} \times \mathbb{P}^n$$

that is, the k-plane Λ sitting inside $\mathbb{P}^n$. Then we obtain a family of k-planes over B, simply by taking the fibre square,

$$\begin{tikzcd} \Sigma' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma \arrow[d] \\ B \arrow[r, "f"] & \mathbb{G}(k,n). \end{tikzcd}$$

For this reason, we call the family $\Sigma \longrightarrow \mathbb{G}(k,n)$ the universal family. Note that we can reverse this process. Suppose that $\mathbb{G}(k,n)$ represents the functor F. By considering the identity morphism $\mathbb{G}(k,n) \longrightarrow \mathbb{G}(k,n)$, we get a family $\Sigma \longrightarrow \mathbb{G}(k,n)$, which is universal, in the sense that to obtain any other family, over any other base, we simply pullback Σ along the morphism $f\colon B \longrightarrow \mathbb{G}(k,n)$, whose existence is guaranteed by the universal property of $\mathbb{G}(k,n)$ (that is, that it represents the functor). To summarise the previous discussion: to prove (3.10) it suffices to construct the natural family over $\mathbb{G}(k,n)$ and prove that it is the universal family.

We won't prove (3.10) here. We will simply observe that the natural family exists, without proving that it is in fact also universal. Recall that the Grasmmannian is by definition the set of all rank one elements ω of $\bigwedge^{k+1} K^{n+1}$. The universal family is then the set

$$\{ (\omega, v) \in \bigwedge^{k+1} V \times V \,|\, \omega \wedge v = 0 \},$$

which is easily seen to be algebraic.

Before we go deeper into the geometry of the Grassmannian, it is interesting to note that the space of conics satisfies the same universal property. Suppose $\mathbb{P}^2 = \mathbb{P}(V)$. Then $\mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}^2(V^*))$ represents the functor G which assigns to every variety B, the set of all (flat) families of conics in $\mathbb{P}^2$, over B. As before the key thing is to show that the natural family of conics in $\mathbb{P}^2$ over $\mathbb{P}^5$, is in fact a universal family. As before we won't show that the natural family is universal, but we observe that the natural family does exist. Indeed,

$$aX^2 + bY^2 + cZ^2 + dYZ + eXZ + fYZ,$$

is bihomogeneous of degree $(1, 2)$ and cuts out the natural family.

Using the diagram,

$$\begin{tikzcd} \Sigma \arrow[r, "q"] \arrow[d, "p"'] & \mathbb{P}^n \\ \mathbb{G}(k, n), & \end{tikzcd}$$

one can make some interesting constructions. For example, suppose we are given a closed subset $X \subset \mathbb{P}^n$. Then $p(q^{-1}(X))$ is a closed subvariety of $\mathbb{G}(k, n)$, consisting of all k-planes in $\mathbb{P}^n$ which intersect X. The first interesting case is that of a curve C in $\mathbb{P}^3$. In this case the general line does not meet the curve C. In fact we get a codimension one subvariety of $\mathbb{G}(1, 3)$. Conversely suppose we are given a closed subvariety Φ of $\mathbb{G}(k, n)$. Then $q(p^{-1}(\Phi))$ is a closed subvariety of $\mathbb{P}^n$, equal to

$$X = \bigcup_{\Lambda \in \Phi} \Lambda.$$

Note that X has the interesting property that through every point of X there passes a k-plane. Classically such varieties are called **scrolls**. Perhaps the first interesting example of a scroll is the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$.

Let us give some more constructions of scrolls. Suppose that we are given two subvarieties X and Y of $\mathbb{P}^n$. Define a rational map

$$\phi\colon X \times Y \dashrightarrow \mathbb{G}(1, n),$$

by sending

$$([v], [w]) \longrightarrow [v \wedge w].$$

The subvariety in $\mathbb{P}^n$, corresponding to the image, is called the **join**. It is the closure of the union of all lines obtained by taking the span of a point of X and a point of Y. Note that ϕ is a morphism if X and Y are disjoint and in this case we don't need to take the closure. If we take $X = Y$, then we get the **secant variety of** X, which is the closure of all the lines which join two points of X.

Suppose that we are given a morphism $f\colon X \longrightarrow Y$, with the property that there is a point $x \in X$ such that $f(x) \neq x$. Consider the morphism
$$\psi\colon X \longrightarrow \mathbb{G}(1, n),$$
which is the composition of
$$X \longrightarrow X \times Y \qquad \text{given by} \qquad x \longrightarrow (x, f(x)),$$
and the morphism ϕ above. As before this gives us a scroll in $\mathbb{P}^n$, by taking the image. Note that all of this generalises to products of k varieties.

Definition 3.11. *Pick complimentary linear spaces $\Lambda_1, \Lambda_2, \dots, \Lambda_k$ of dimensions $n_1, n_2, \dots, n_k$ in $\mathbb{P}^n$, where*
$$n + 1 = \sum_i (n_i + 1).$$
Pick rational normal curves $C_i \subset \Lambda_i$ in and pick identifications
$$\phi_i\colon \mathbb{P}^1 \longrightarrow C_i.$$
Let
$$X = \bigcup_{p \in \mathbb{P}^1} \langle \phi_1(p), \phi_2(p), \dots, \phi_k(p) \rangle.$$
Then X is called a ***rational normal scroll****.*

It is interesting to give some examples. Suppose that we pick two skew lines l and m in $\mathbb{P}^3$. Then we get a surface in $\mathbb{P}^3$, swept out by lines, meeting l and m. Suppose we pick coordinates such that $l = V(X, Y)$ and $m = V(Z, W)$. Identify $(0, 0, a, b)$ with $(a, b, 0, 0)$. Then it is not hard to see that we get the surface $V(XW - YZ)$.

The next case is when we take a line and a complimentary plane in $\mathbb{P}^4$. The resulting surface in $\mathbb{P}^4$ is called the cubic scroll.

Let us now investigate how to work with the Grassmannian in the case of schemes. As in the case of affine and projective space we can define a scheme over $\operatorname{Spec} \mathbb{Z}$ and use this scheme to define the Grassmannian over any base scheme. In fact the equations defining the Grassmannian over an algebraically closed field have integral coefficients (better still, presumably 0 and ± 1) and this defines the Grassmannian as a closed subscheme of $\mathbb{P}^N_{\mathbb{Z}}$. However this somehow begs the question; what role does the Grassmannian play over an arbitrary base scheme S?. We want to extend the functor F, which is a priori defined only as a functor from varieties over K to $(\underline{\text{Sets}})$, to a functor from the category of schemes over S to the category $(\underline{\text{Sets}})$. To answer this question, we need to decide what we mean by a family of k-planes in $\mathbb{P}^n_S$. It turns out to be easier to answer what it means to have a family of vector subspaces of dimension $k+1$.