12. Singularities

It is the aim of this section to develop some of the theory and practice of the classification of singularities in algebraic geometry. If one want to classify singularities, then this is clearly a local problem. Unfortunately the Zariski topology is very weak, and the property of being local in the Zariski does not satisfactorily capture the correct notion of classification. In general the correct approach is to work with the formal completion. Since this is somewhat technical, we work instead over $\mathbb{C}$, and work locally analytically.

The most basic invariant of a singular point is the dimension of the Zariski tangent space.

Definition 12.1. *Let (X, p) be a germ of a singularity. The* ***embedding dimension*** *is the dimension of the Zariski tangent space of X at p.*

As the name might suggest, we have the following characterisation of the embedding dimension.

Lemma 12.2. *Let (X, p) be a germ of a singularity. The* ***embedding dimension*** *is equal to the smallest dimension of any smooth germ (M, q) such that X embeds in M.*

Proof. Let k be the embedding dimension of X, and suppose that $X \subset M$, where M is smooth. As $T_pX \subset T_pM$, and the dimension of M is equal to the dimension of T_pM, it is clear that the dimension of M is at least k.

Now consider embedding X into a smooth germ N and then projecting down to a smaller subspace M. Clearly we can always choose the projection to be an embedding of the Zariski tangent space to X at p, provided the dimension of M is at least k. Since the property that df is an isomorphism of Zariski tangent spaces is a local condition, it follows that possibly passing to a smaller open subset, we may assume that projection down to M induces an isomorphism of Zariski tangent spaces. But then the projection map is an isomorphism. $\square$

Definition 12.3. *We will say that X has a* ***hypersurface singularity*** *if the embedding dimension is one more than the dimension of X; we will say that a curve singularity is* ***planar*** *if it is a hypersurface singularity.*

Let $(X, p) \subset (M, p)$ be a hypersurface singularity. Pick coordinates $x_1, x_2, \dots, x_n$ on M and suppose that f defines X. Let m be the maximal ideal of M at p. The ***multiplicity of*** *X* ***at*** *p is equal to the smallest integer μ such that $f \in m^\mu$.*

Given X, let Y be the singularity given by $x^2 + f$, where x is a new variable, so that Y is a hypersurface singularity of dimension one more than X. Any singularity obtained by successively replacing f by $x^2 + f$ will be called a ***suspension of*** *X.*

*We say that X has type A_n**-singularities** if X is defined by the suspension of x^{n+1}. We say that X has type D_n**-singularities**, for $n \geq 4$, if X is defined by the suspension of $x^2y + y^{n-1}$. We say that X has a type E_6**-singularity**, if X is defined by the suspension of x^3+y^4, a type E_7**-singularity**, if X is defined by the suspension of $x^3 + xy^3$, and a type E_8**-singularity**, if X is defined by the suspension of x^3+y^5.*

Note that the multiplicity of X is independent of the choice of coordinates and that a hypersurface is smooth iff the multiplicity is one. Note that the multiplicity is upper semi-continuous in families.

There are a couple of basic results about power series that we will use time and again. First some basic notation. We say that a monomial m appears in f and write $m \in f$ if the coefficient of m in f is non-zero.

Lemma 12.4. *Let $f \in \mathbb{C}\{x_1, x_2, \dots, x_n\}$ be the germ of an analytic function.*

1. *If f has non-zero constant term then f is invertible and we may take nth roots.*
2. *If we write $f = ax_n^k + \dots$, where dots indicate terms divisible by x_n^k of higher degree and $a \neq 0$, then we may change coordinates so that $f = x_n^k$.*
3. *If $f = ux_n^k + \dots$, where $\dots$ indicate terms other than x_n^k and u is not in the maximal ideal, then we may change coordinates so that $f = x_n^k + f_{n-2}x_n^{k-2} + \dots + f_0$, where f_i are analytic functions in the first $n - 1$ variables.*

Proof. (1) is well-known. Consider (2). By assumption we may write $f = ax_n^k + x_n^k g$, where g is an analytic function lying in the maximal ideal. In this case $f = x_n^k(a + g) = x_n^k u$, where by (1) u is a unit. In this case, also by (1), there is an analytic function v such that $v^k = u$. Replacing x_n by vx_n, f now has the correct form. This is (2).

Finally consider (3). Clearly we may expand f as

$$f = \sum_i f_i x_n^i,$$

where f_i are power series in the first $n - 1$ variables. By assumption f_k is a unit. As before, we may then assume that $f_k = 1$. By (2) we may assume that $f_i = 0$ for $i > k$. Completing the nth power we may assume that $f_{k-1} = 0$. Now f has the required form. $\square$

Definition-Lemma 12.5. *Let X be a hypersurface singularity of multiplicity μ. Then we may choose coordinates $x_1, x_2, \dots, x_n$ such that X is given by*

$$x_n^{\mu} + f_{\mu-2}x_n^{\mu-2} + \cdots + f_0,$$

where f_i are analytic functions of the first $n-1$ variables. Any such polynomial is called a ***Weierstrass polynomial****.*

Proof. By assumption f_μ is non-zero. Possibly changing coordinates, we may assume that $x_n^\mu \in f$. The result is now an easy consequence of (12.4). □

Lemma 12.6. *A planar curve singularity has multiplicity two iff it is of type A_n.*

Proof. After putting f into Weierstrass form, the result becomes easy. □

It is interesting to see what happens for small values of n. If $n = 1$, so that $f = y^2 + x^2$, then we have a **node**. This corresponds to two smooth curves with distinct tangent directions. If $n = 2$, then $f = y^2 + x^3$, then we have a **cusp**. In the case $n = 3$, we have $y^2 + x^4$, this represents two smooth curves which are tangent. We call this a **tacnode**. The case $n = 4$ is called a **ramphoid cusp**, $n = 5$ an **oscnode**, and $n = 6$ a **hyper-ramphoid cusp** and so on.

Definition 12.7. *Let C be a planar singularity of order μ. We say that C is* ***ordinary*** *if when we write $f = f_\mu + \dots$, where dots indicate higher order terms, then f_μ factors into μ distinct linear factors.*

It is not hard to show that that if C is ordinary, we may always choose coordinates so that $f = f_\mu$.

Definition 12.8. *Let X be a singular variety, a subset of $\mathbb{A}^n$. The* ***tangent cone*** *of X at a point p is the intersection of the strict transform of X with the exceptional divisor.*

If X is a hypersurface singularity, then the tangent cone is given by $f_\mu = 0$ a subset of $\mathbb{P}^{n-1} \simeq E$.

Example 12.9. *Consider ordinary planar curve singularities of multiplicity four. Then each linear factor defines an element of $\mathbb{P}^1$. But four unordered points in $\mathbb{P}^1$ have moduli (the j-invariant). Thus there is a one dimensional family of non-isomorphic planar curve singularities of multiplicity four. Indeed, since one can always choose the first three points to be 0, 1 and ∞, we can write*

$$f = xy(x-y)(x-\lambda y).$$

Definition 12.10. *Let (X,p) be the germ of a singularity. A **deformation** of X is a triple (π,σ,i), where $\pi\colon \mathcal{X}\longrightarrow B$ is a morphism, σ is a **section** of π (that is, $\pi\circ\sigma$ is the identity) such that for every $t\in B$ the pair $(X_t,\sigma(t))$ is a germ of a singularity and i is an **isomorphism** of the pair (X,p) and $(X_0,\sigma(0))$, the central fibre of π.*

In practice, it is customary to drop σ and i and refer to a deformation using only π. Note that since the multiplicity is upper semi-continuous in families, it follows that the multiplicity can only go down under deformation.

In other words, a deformation is to the germ of a singularity, as a family is to a variety. As such one might hope that there exists universal deformations, as there exists universal families. Equivalently, one might hope to write down the obvious functor and hope that there is a space which represents this functor. Unfortunately this is not so; the problem is that the central fibre might have more automorphisms, than the typical fibre (and this why we are careful to specify the isomorphism of the central fibre with the space to be deformed). Instead, the best we can hope for is

Definition 12.11. *Let (X,p) be a germ of a singularity. We say that a deformation π of X is **versal** if for every other deformation ψ there is a morphism $B'\longrightarrow B$ such that ψ is pulled back from π in the obvious way.*

Note that we do not require uniqueness of the versal family, and in fact we cannot, since if there is an automorphism of the central fibre that does not lift to the whole deformation space, for example if it does not lift to every fibre, then we get a different deformation, simply by composing with this automorphism (that is, we change the isomorphism i).

Fortunately, versal deformation spaces are easy to write down.

Definition 12.12. *Let X be a hypersurface singularity, defined by the equation $f=0$. Let*

$$T^1_f=\frac{\mathbb{C}[x_1,x_2,\dots,x_n]}{\langle f,\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\dots\frac{\partial f}{\partial x_n}\rangle}.$$

Theorem 12.13. *Let X be an isolated hypersurface singularity. Pick holomorphic functions $g_1,g_2,\dots,g_k$ such that their images in T^1_f form a basis. Then the deformation given by*

$$f_t=f+\sum_i t_ig_i,$$

where t_i are coordinates on the germ $(\mathbb{C}^k,0)$ is a versal deformation.

Another way to state (12.13), is that T^1_f is the Zariski tangent space to the versal deformation space. We will also need the following basic fact.

Lemma 12.14. *Let $\pi\colon \mathcal{X} \longrightarrow B$ be a versal deformation space, and let B' be a general subvariety of B. Then the restriction of π to B' defines a versal deformation space of the general point of B'.*

It is interesting to see what happens in a series of examples. Suppose we start with planar singularities. The simplest is an A_1-singularity. In this case

$$f = x^2 + y^2 \qquad \text{so that} \qquad \frac{\partial f}{\partial x} = 2x \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 2y.$$

Thus

$$T^1_f = \frac{\mathbb{C}[x,y]}{\langle x, y\rangle}$$

and for g_1 we take 1. Thus the versal deformation space of a node is given as

$$f = x^2 + y^2 + t.$$

In other words, the only thing that we can do with a node is smooth it. Now consider what happens in the case of an A_n-singularity. In this case the derivatives are $(n+1)x^n$ and $2y$, so that we make take $g_i = x^i$, $i = 0 \dots n-1$. Thus the versal deformation space has dimension n, and it is given by

$$y^2 + x^n + t_0 + t_1 x + t_2 x^2 + \dots + t_{n-1}x^{n-1}.$$

For example consider the case of a cusp. In this case the versal deformation space has dimension two, and it is given by

$$y^2 + x^3 + ax + b.$$

where a and b are coordinates on the base. The point is that now we have two completely different one dimensional families. Either we can smooth the cusp, or we can partially smooth it to a node. In fact the locus of nodes forms a curve in the base.

Note that $y^2 = x^3 + ax + b$ is singular iff the polynomial $x^3 + ax + b$ has a double root. But then the singular locus is given by the discriminant, that is $4b^3 + 27a^2$, so that this locus is not smooth.

Similarly, it is not hard to see that the locus of A_n-singularities contains loci corresponding to the A_k-singularities, for $k \leq n$. In fact this locus will have codimension $n-k$.

In fact the converse is true, that is, one can only deform an A_n-singularity to an A_k singularity, for $k \leq n$. Compare this with the case of an ordinary four-fold point. Suppose that we start with $x^4 + y^4$.

Then the derivatives are $4x^3$ and $4y^3$ and so we can take for g_i, 1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2. In this case, there is a one dimensional locus corresponding to all other (nearby) ordinary four-fold singularities, given by $x^4 + y^4 + tx^2y^2$. In other words, there are infinitely many non-isomorphic germs in the versal deformation space.

It is interesting to look at some of these ideas from the point of view of blowing up and resolution of singularities.

Definition 12.15. *Let X be a variety and let $D = \sum D_i$ be a divisor, the sum of distinct prime divisors. We say that the pair (X, D) has normal crossings if X is smooth and locally about every point, the pair (X, D) is equivalent to $\mathbb{C}^n$ union some of the coordinate hyperplanes.*

*A **resolution of singularities** for X is a birational map $\pi\colon Y \longrightarrow X$ with the following properties.*

(1) *π is an isomorphism over the smooth locus of X,*
(2) *Y is smooth,*
(3) *the exceptional locus is a divisor with normal crossings.*

To date there is only one known way to resolve singularities (at this level of generality) and that is to embed X into a smooth variety M and then carefully choose an appropriate sequence of blow ups, at each stage blowing up M and replacing X by its strict transform. In this case we want the exceptional locus of $\psi\colon N \longrightarrow M$ to intersect the strict transform Y of X as transversally as possible. For example if X has a hypersurface singularity, then we want $Y + E$, where E is the exceptional locus, to have normal crossings.

We have already seen some examples of this. Perhaps the easiest example is the case of a nodal curve. In this case C sits inside a smooth surface M, and we simply blow up the singular point of C. At this point C is smooth and meets the exceptional locus smoothly in two points, so that the pair $C + E$ does have normal crossings.

Now suppose that we take a curve with a cusp. Pick local coordinates so that we have $y^2 + x^3$. Blowing up once, we have already seen that C becomes smooth. However C is tangent to the exceptional locus. If we blow up, then the strict transform of C intersects the points where the two exceptional divisors intersect. Thus it is necessary to blow up once more to achieve normal crossings.

It is interesting to see what happens for an ordinary singularity. In this case we have seen that we may choose coordinates so that f is homogeneous. Thus f factors into μ distinct linear factors. Now each of those factors corresponds to a point of the exceptional locus and in fact when we blow up then C is smooth and meets the exceptional divisor at μ points. At this point $C + E$ has normal crossings.

For example, if the multiplicity is four, then C meets E in four points, and we get our j-invariant directly.

Theorem 12.16. *Let C be a planar curve singularity.*

Then the versal deformation space of C contains only finitely many isomorphism types iff C is one of A_n, D_n, E_6, E_7 or E_8.

Proof. Let f be a defining equation for C. Let us show that if there are only finitely many isomorphism types in the versal deformation space, then C must be one of the ADE-singularities. Suppose that the multiplicity of f is at least four. Then we may deform f to an ordinary multiplicity four singularity. But then there are infinitely many non-isomorphic singularities in the versal deformation space. On the other hand, if the multiplicity is two, then by (12.6) C must have type A_n.

Thus we may suppose that f has multiplicity three. Consider f_3. This factors into three linear factors. There are three cases; the three factors are distinct; there are two distinct factors, there is one.

Suppose that there are at least two distinct factors. Then there is a factor which occurs only once. We may assume that this factor is y. Since the multiplicity is three, in fact y must divide f, so that we may write

$$f = h \cdot y,$$

where h only depends on x and y. Now h has multiplicity two and its rank two part is not divisible by y. It follows that there is a change of variable, so that $h = x^2 + y^n$, where $n \geq 2$, which change of variable does not change y. But then $f = x^2y + y^{n+1}$ and we have a singularity of type D (more precisely, a D_{n+3}-singularity).

This final case is when $f_3 = y^3$, so that $f = y_3 + g$ and the multiplicity of g is at least four. Putting f into Weierstrass form, once again, we may assume that $f = y^3 + yg + h$, where g and h only involve x. Thus f can be put in the form $y^3 + ayx^k + x^l + \dots$, where the dots indicate higher powers of x, a is either zero or one and $k < l$. If $l = 4$, it follows that $k = 3$ so that completing the square we may assume that $a = 0$. In this case, it is not hard to show that we can choose coordinates so that $f = y^3 + x^4$ and we have an E_6-singularity. If $x^3y \in f$ and $l > 4$ then with some manipulation we can put f into the form $y^3 + x^3y$, so that we have an E_7-singularity; similarly if $y^5 \in f$ but we have no lower terms, then we have an E_8-singularity. Otherwise we may assume that $l > 5$ and that $k > 3$. In this case, we may as well assume that $f = y^3 + \lambda yx^4 = y(y^2 + \lambda x^4)$, which represents three smooth curves which are tangent. Suppose we blow up once; we get four curves passing through one point, the strict transform of the three tangent curves and the exceptional divisor. If we blow up the point, we get four curves intersecting the new exceptional divisor and the four points of intersection with the new exceptional gives one dimension of moduli (the j-invariant, which varies as we vary λ).

Now let us consider the converse problem, to show that there are only finitely many isomorphism types in the versal deformation space. Clearly it suffices to prove that we can only deform an ADE-singularity, to an ADE-singularity. This is clear for A_n-singularities, since under deformation the multiplicity can only go down.

Now consider the case of a D_n-singularity. We only need to consider deformations that preserve the multiplicity. In this case, the deformations of f_3 can only increase the number of distinct linear factors, and we cannot lose a term of the form y^k. Thus the deformation of an D_n-singularity is either a D_k-singularity, for some $k \leq n$ or an A_n-singularity.

Finally consider the three exceptional cases. Suppose we start with x^3+y^4. Then the only possible deformation which fixes the multiplicity, deforms to a singularity of type D_n, $n \leq 5$. Now suppose we start with $y^3 + x^3y$. Again we can only pick up a term of the form x^4 or increase the number of linear factors. Similarly for an E_8-singularity. $\square$

Here is a way to restate (12.16):

Proposition 12.17. *The ADE-singularities are the only singularities, which have multiplicity two and three, and such that after blowing up, the multiplicity of the total transform has multiplicity two or three.*

Proof. It is not hard to check that this is all we have used in the proof of (12.16) to characterise ADE-singularities. $\square$

The are four other obvious ways of creating examples of singularities other than writing down equations. The first is simply to take the cone over a closed subset of $\mathbb{P}^n$. Note that the cone is a degenerate example of the join of two varieties, where one of the two varieties to be joined is a point. Note also that if I is the ideal of $X \subset \mathbb{P}^n$, then I is also the ideal of the cone Y over X, where Y is the closure of the inverse image of X inside K^{n+1}. In particular, the classification of singularities is at least as hard as the classification of varieties. On the other hand, note that the resolution problem for such singularities is in fact easy. If X is smooth, then simply blowing up the vertex, we get a birational map $\pi\colon W \longrightarrow Y$, whose exceptional locus E is a copy of X, where W is smooth. In fact W is a $\mathbb{P}^1$-bundle over X, and E is simply a section of this bundle.

The next is to start with a configuration of divisors and contract them. Unfortunately it is quite hard to characterise which configurations are contractible. The third method is to take a quotient:

Definition 12.18. *Let G be an algebraic group acting on a variety X. We say that Y is a **categorical quotient of** X **by** G if there is a morphism $\pi\colon X \longrightarrow Y$ such that $\pi(g \cdot x) = \pi(x)$ for every $g \in G$, which is universal amongst all such morphisms:*

If $\phi\colon X \longrightarrow Z$ is a morphism such that $\psi(g \cdot x) = \psi(x)$ then there is a unique morphism $\psi\colon Y \longrightarrow Z$ which makes the diagram commute,

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \\ Y \arrow[r, "\psi"] & Z. \end{tikzcd}$$

It is common to denote the categorical quotient by X/G (if it exists at all). Fortunately there is one quite general existence theorem:

Theorem 12.19. *Let $X = \operatorname{Spec} A$ be an affine variety and let G be a finite group acting on X.*

Then the categorical quotient is the affine variety $Y = \operatorname{Spec} A^G$.

Proof. The key fact is that the ring of invariants

$$A^G = \{\, a \in A \mid g \cdot a = a \,\},$$

is a finitely generated k-algebra. □

Note that $Y = X/G$ will in general be a singular variety. It is however $\mathbb{Q}$-factorial, that is, every Weil divisor is $\mathbb{Q}$-Cartier, that is, given by any Weil divisor D, some multiple is Cartier (indeed, $r = |G|$ will do).

The final method is to use toric geometry. We start with the canonical example.

Let σ be the cone spanned by e_2 and $2e_1 - e_2$. The dual cone σ is spanned by f_1 and $f_1 + 2f_2$. Generators for the monoid are f_1, $f_1 + f_2$ and $f_1 + 2f_2$, so that

$$X = U_\sigma = \operatorname{Spec} \mathbb{C}[x, xy, xy^2] = \operatorname{Spec} \mathbb{C}[u, v, w]/\langle v^2 = uw \rangle.$$

Thus we have an A_1-singularity.

It is interesting to see how to resolve this singularity. Suppose we insert the vector e_1; this corresponds to a blow up with exceptional divisor isomorphic to $\mathbb{P}^1$. We get two cones σ_1 and σ_2, one spanned by e_1 and e_2 and the other spanned by e_1 and $2e_1 - e_2$. It follows that the blow up is smooth. Note that X is the cone over a conic; it follows once again that X can be resolved in one step.

Let's make this example a little more complicated. Let's start with the cone spanned by e_2, $re_1 - e_2$. The dual cone is the cone spanned by f_1 and $f_1 + rf_2$. Generators for the monoid are f_1, $f_1 + f_2$, $\dots$, $f_1 + rf_2$. We get

$$U_\sigma = \operatorname{Spec} \mathbb{C}[x, xy, \dots, xy^r] = \operatorname{Spec} \mathbb{C}[u^r, u^{r-1}v, \dots, v^r],$$

where $u^r = x$ and $v = y/x$, which is the cone over a rational normal curve of degree r. Note that the embedding dimension is $r + 1$. Note that this is again resolved in one step by inserting the vector e_1.

At the other extreme, consider the cone spanned by e_2 and $re_1 - (r - 1)e_2$. The dual cone is spanned by f_1 and $(r-1)f_1 + rf_2$. Generators for the monoid are f_1, $(r-1)f_1 + rf_2$ and $f_1 + f_2$. We get

$$U_\sigma = \operatorname{Spec} \mathbb{C}[x, xy, x^{r-1}y^r] = \operatorname{Spec} \mathbb{C}[u, v, w]/\langle v^r = uw\rangle,$$

which is an A_{r-1}-singularity. If we insert the vector e_1 then we the resulting blow up has two affine pieces. One is smooth, corresponding to the cone spanned by e_1 and e_2 and the other is the cone given by e_1 and $re_1 - (r-1)e_2$. Switching the sign of e_2 we get e_1, $re_1 + (r-1)e_2$. Switching e_1 and e_2 we get e_2 and $(r-1)e_1 - re_2$. Replacing e_1 by $e_1 - 2e_2$ we get e_2 and $(r-1)e_1 - (r-2)e_2$ which as we have already seen is an A_{r-2}-singularity. Thus an A_r-singularity takes r-steps to resolve. On the resolution we get a chain of r-copies of $\mathbb{P}^1$.

More generally, we could consider the cone spanned by e_2 and $re_1 - ae_2$, where $0 < a < r$, is coprime to r. However the best way to proceed, is to look at all of this a different way.

We start with an example. The cyclic group $G = \mathbb{Z}_r$ acts on $\mathbb{C}^2$ via

$$(u, v) \longrightarrow (\omega u, \omega v),$$

where ω is a primitive rth root of unity. In this case the ring of invariants is precisely

$$\mathbb{C}[u, v]^G = \mathbb{C}[u^r, u^{r-1}v, \dots, v^r].$$

To see this using the toric structure, let $N' \subset N$ be the sublattice spanned by $e'_2 = e_2$ and $e'_1 = re_1 - e_2$. Then the cone σ' spanned by the same vectors e_2 and $re_1 - e_2$ now corresponds to a smooth toric variety. The dual lattice M' is an overlattice of M.

Thinking this way, we should make a basis for N' the standard vectors e'_1 and e'_2. The overlattice N is spanned by N' and the vector e_1 in the old coordinates. As

$$e_1 = 1/r(re_1 - e_2) + 1/re_2,$$

in the old coordinates, in the new coordinates we have that N' is spanned by e'_1, e'_2 and $1/r(e'_1 + e'_2)$. If we insert this vector, we get a basis for the lattice. If the dual lattice M' is the overlattice spanned by f_1' and f_2' then M is the sublattice spanned by all $af_1' + bf_2'$ such that $a+b$ is divisible by r.

In the other example, where we started with e_2 and $re_1 - (r-1)e_2$, then

$$e_1 = 1/r(re_1 - (r-1)e_2) + (r-1)/re_2.$$

So N is the lattice spanned by e_1, e_2 and $1/re_1 + (r-1)/re_2$. This suggests we should look at the action

$$(x, y) \longrightarrow (\omega x, \omega^{r-1} y) = (\omega x, \omega^{-1} y),$$

Indeed, the ring of invariants is $u = x^r$, $w = y^r$ and $v = xy$ and $v^r = uw$, as expected.

More generally still, for the action

$$(x, y) \longrightarrow (\omega x, \omega^a y),$$

we should look at the lattice N spaned by the standard lattice and the vector $1/r(1, a)$. Inserting this vector, gives two cones, one spanned by e_1, $1/r(1, a)$ and the other spanned by $1/r(1, a)$ and e_2. The second one is smooth. For the first, let us make the two vectors $1/r(1, a)$ and e_1 the standard generators for the lattice. As

$$(0, 1) = r/a(1/r, a/r) - 1/a(0, 1),$$

we then have the overlattice generated by $(-1/a, r/a)$. Now

$$r/a = k - b/a,$$

for some unique $0 \leq b < a$. So we get a singularity of type $1/a(1, b)$. Note that resolving the singularity corresponds to computing a continued fraction. The significance of k is the self-intersection of exceptional divisor (on the minimal resolution).

So the resolution graph of any cyclic surface singularity is a chain of $\mathbb{P}^1$'s. Singularities of type A_r correspond to a chain of r such curves, where each curve has self-intersection -2. In fact it is not hard to prove:

Theorem 12.20. *Let $S = \mathbb{C}^2/G$ be a two dimensional quotient singularity. Then $G \subset \mathrm{GL}(2, \mathbb{C})$ and there are three possibilities:*

(1) *G is cyclic and the dual graph of the (minimal) resolution corresponds to the Dynkin diagram A_n. The action is $(x, y) \longrightarrow (\omega x, \omega^a y)$, where ω and ω^a is both primtive roots of unity. S is isomorphic to a toric surface.*
(2) *G is a dihedral group and the dual graph corresponds to the Dynkin diagram for D_n, $n \geq 4$.*

(3) *G is one of three exceptional groups and the dual graph is the Dynkin diagram for E_6, E_7 or E_8.*

If in addition $G \subset \mathrm{SL}(2, \mathbb{C})$ then S has an ADE-singularity and the self-intersections of the exceptional curves are all -2.

More generally suppose that $\sigma \subset N \simeq \mathbb{Z}^n$ is a simplicial cone. As before let $N' \subset N$ be the sublattice spanned by the primitive generators of σ. Let $M \subset M'$ be the corresponding overlattice. Then there is a natural pairing

$$N/N' \times M'/M \longrightarrow \mathbb{Q}/\mathbb{Z}$$

This makes M the invariant sublattice of M', under the action of the finite abelian group $G = N/N'$ and under this action it is not hard to see that

$$A_\sigma = (A_{\sigma'})^G.$$

Note that G is a product of at most $n-1$ cyclic factors.

Let me end by talking a little about the problem of resolution of singularities. At it most basic we are given a finitely generated field extension K/k and we would like to find a smooth projective variety X over k with function field K.

Theorem 12.21. *Let X be a smooth projective variety and let $\mathcal{O}_X(1)$ be a very ample line bundle. Suppose that $X \subset \mathbb{P}^n$ has degree d.*

Then

$$h^0(X, \mathcal{O}_X(m)) = \frac{dm^n}{n!} + ...,$$

is a polynomial of degree n, for m large enough, with the given leading term.

Proof. Let Y be a hyperplane section. The trick is to compute $\chi(X, \mathcal{O}_X(m))$ by looking at the exact sequence

$$0 \longrightarrow \mathcal{O}_X(m-1) \longrightarrow \mathcal{O}_X(m) \longrightarrow \mathcal{O}_Y(m) \longrightarrow 0.$$

The Euler characteristic is additive so that

$$\chi(X, \mathcal{O}_X(m)) - \chi(X, \mathcal{O}_X(m-1)) = \chi(Y, \mathcal{O}_Y(m)).$$

By an easy induction, it follows that $\chi(X, \mathcal{O}_X(m))$ is a polynomial of degree n, with the given leading term. Now apply Serre vanishing. □

Definition 12.22. *Let $X \subset M$ be a subvariety of a smooth variety. The **multiplicity of** X **at** $p \in M$ is the smallest μ such that $\mathcal{I}_p \subset \mathfrak{m}^\mu$ where $\mathfrak{m}$ is the maximal ideal of M at p in $\mathcal{O}_{M,p}$ and $\mathcal{I}$ is ideal sheaf of X in M.*

The method of Albanese is to start with $X \subset \mathbb{P}^n$. Now re-embed X by the very ample line bundle $\mathcal{O}_X(m)$. The degree of the image is dm^n inside $\mathbb{P}^r$, where r is roughly $dm^n/n!$. Now suppose that there is a point p of multiplicity μ. If we project from p then we drop r by 1 and the degree by μ. So if we take m sufficiently large and always project from a point of highest multiplicity then we can also reduce to the case when the multiplicity is at most $n!$. Unfortunately it seems impossible to improve this bound.

Another intriguing method was proposed by Nash:

Definition 12.23. *Let $X \subset \mathbb{P}^N$ be a quasi-projective variety of dimension n. The **Gauss map** is the rational map*

$$X \longrightarrow \mathbb{G}(n, N) \qquad \text{given by} \qquad x \longrightarrow T_x X,$$

which sends a point to its (projective) tangent space.

Conjecture 12.24. *We can always resolve any variety by successively taking the Nash blow up and normalising.*

Despite the very appealing nature of this conjecture (consider for example the case of curves) we only know (12.24) in very special cases (the result for toric varieties is about six months old).

If X is a toric variety there is a pretty simple method to resolve singularities. First subdivide the cone until X is simplicial. It is not too hard to argue that one can resolve any simplicial toric variety (one keeps track of a simple invariant).

In general the only known method to find a strong resolution of singularities (only touch the smooth locus) goes back to Hironaka. The idea is embed X into a smooth variety M and choose a sequence of smooth blow ups in M. The problem reduces to two (closely related) parts. Determine the locus to blow up at every step and find some invariant which goes down if we blow this locus up. Forty years after Hironaka's original proof, we know now the only invariant we need to keep track of is the multiplicity.

Unfortunately it is also clear that we need to be quite careful how to choose the locus to blow up. For example consider

$$z^2 - x^3 y^3.$$

The singular locus consists of the x and y axis. If we blow up either axis it is clear that we are making progress (generically along the y-axis we have $z^2 - x^3$, which is resolved in three steps by blowing up the origin). But we are not allowed to blow up an axis. The problem is that this might only be the local analytic picture. We might globally have the singular locus be a nodal cubic (for example). So our resolution process must respect all local isomorphisms. There is an obvious x-y symmetry and so we cannot blow up only one axis. The only possible locus we could blow up which is in the singular locus is the origin. On the blow up we have coordinates $(x, y, z) \times [A : B : C]$, and equations expressing the rule $[x : y : z] = [A : B : C]$. On the coordinate patch $A \neq 0$ we have $y = bx$, $z = cx$ so that

$$z^2 - x^3y^3 = c^2x^2 - b^3x^6 = x^2(c^2 - b^3x^4).$$

Changing variables we have $z^2 - x^3y^4$ which is surely worse than before. The key thing is that the singular locus is given by $c = b = 0$ and $c = x = 0$. The first singular locus we created ourselves and so we know that we are allowed to blow up $c = b = 0$ and now we can desingularise.

So the blow up process must keep track of the sequence of blow ups.