HWK #6, DUE WEDNESDAY 3/31

1. Let X be a smooth variety of dimension n over an algebraically closed field k of characteristic zero. What is the dimension of the space of curvilinear schemes of length l? (*Hint: You may assume the naive dimension count is correct.*)
2. Let k be an algebraically closed field of characteristic zero. Fix integers p, q and n, and let $\mathfrak{m} \subset k[x_1, x_2, \dots, x_n]$ be the maximal ideal of the origin in $\mathbb{A}^n_k$. Let $A = k[x_1, x_2, \dots, x_n]/I$ where $I \triangleleft k[x_1, x_2, \dots, x_n]$ is any ideal such that

$$\mathfrak{m}^{p+1} \subset I \subset \mathfrak{m}^p,$$

and the quotient $W = I/\mathfrak{m}^{p+1} \subset \mathfrak{m}^p/\mathfrak{m}^{p+1} = V$ is a vector subspace of dimension q.
(i) What is the length of $z = \operatorname{Spec} A$?
(ii) Show that there is a correspondence between ideals I and subvector spaces $W \subset V$.
(iii) What is the dimension of the space of all such zero dimensional subschemes $z \subset \mathbb{A}^n_k$ supported at the origin? (*Hint: Use the Grassmannian. You may assume the naive dimension count is correct.*)
3. Comparing your answers to (1) and (2) show that the space of zero dimensional schemes in $\mathbb{A}^n$ is not the closure of the locus of curvilinear schemes (*Hint: It is for $n = 2$ (don't prove this; it is hard!) but it is not for $n = 3$*).
4. **An affine bundle of dimension** n is a morphism $\pi\colon X \longrightarrow Y$ of schemes such that there is a open cover $\{U_i\}$ of Y, isomorphisms

$$\phi_i\colon \pi^{-1}(U_i) \longrightarrow \mathbb{A}^n_{U_i},$$

for each U_i, where moreover the transition functions

$$g_{ij} = \phi_j \circ \phi_i^{-1}\colon \mathbb{A}^n_{U_{ij}} \longrightarrow \mathbb{A}^n_{U_{ij}},$$

are affine linear (same as linear automorphism, but where we also allow translation, $x \longrightarrow Ax + b$). If π has a section, so that the fibres are vector spaces and (we can arrange that) the transition functions are linear (that is they fix the section), then we say that π is a **vector bundle of rank** n.
(i) Let $\mathcal{H}^l_0$ be the punctual Hilbert scheme of length l zero dimensional schemes in $\mathbb{A}^n_k$ supported at the origin. Let $\mathcal{C}^l_0$ be the locally closed subset (with the reduced induced structure) of curvilinear schemes. Show that $\mathcal{C}^2_0$ is a copy of $\mathbb{P}^{n-1}_k$.

(ii) Show that the natural morphism

$$\pi \colon \mathcal{C}_0^3 \longrightarrow \mathcal{C}_0^2,$$

realises $\mathcal{C}_0^3$ as an affine bundle of dimension $n-1$ over $\mathcal{C}_0^2 = \mathbb{P}^{n-1}$.
(iii) Show that π has a section, and that with this section π becomes a vector bundle of rank $n-1$ over $\mathbb{P}^{n-1}$.