HWK #6, DUE WEDNESDAY 3/31

1. Let $X$ be a smooth variety of dimension $n$ over an algebraically closed field $k$ of characteristic zero. What is the dimension of the space of curvilinear schemes of length $l$? (Hint: You may assume the naive dimension count is correct.)

2. Let $k$ be an algebraically closed field of characteristic zero. Fix integers $p$, $q$, and $n$, and let $m \subset k[x_1, x_2, \ldots, x_n]$ be the maximal ideal of the origin in $\mathbb{A}^n_k$. Let $A = k[x_1, x_2, \ldots, x_n]/I$ where $I \triangleleft k[x_1, x_2, \ldots, x_n]$ is any ideal such that $m^{p+1} \subset I \subset m^p$, and the quotient $W = I/m^{p+1} \subset m^p/m^{p+1} = V$ is a vector subspace of dimension $q$.
   (i) What is the length of $z = \text{Spec } A$?
   (ii) Show that there is a correspondence between ideals $I$ and subvector spaces $W \subset V$.
   (iii) What is the dimension of the space of all such zero dimensional subschemes $z \subset \mathbb{A}^n_k$ supported at the origin? (Hint: Use the Grassmannian. You may assume the naive dimension count is correct.)

3. Comparing your answers to (1) and (2) show that the space of zero dimensional schemes in $\mathbb{A}^n_k$ is not the closure of the locus of curvilinear schemes (Hint: It is for $n = 2$ (don’t prove this; it is hard!) but it is not for $n = 3$).

4. An affine bundle of dimension $n$ is a morphism $\pi: X \rightarrow Y$ of schemes such that there is a open cover $\{U_i\}$ of $Y$, isomorphisms
   \[ \phi_i: \pi^{-1}(U_i) \rightarrow \mathbb{A}^n_{U_i}, \]
   for each $U_i$, where moreover the transition functions
   \[ g_{ij} = \phi_j \circ \phi_i^{-1}: \mathbb{A}^n_{U_{ij}} \rightarrow \mathbb{A}^n_{U_{ij}}, \]
   are affine linear (same as linear automorphism, but where we also allow translation, $x \rightarrow Ax + b$). If $\pi$ has a section, so that the fibres are vector spaces and (we can arrange that) the transition functions are linear (that is they fix the section), then we say that $\pi$ is a vector bundle of rank $n$.
   (i) Let $\mathcal{H}^l_0$ be the punctual Hilbert scheme of length $l$ zero dimensional schemes in $\mathbb{A}^n_k$ supported at the origin. Let $\mathcal{C}_0^0$ be the locally closed subset (with the reduced induced structure) of curvilinear schemes. Show that $\mathcal{C}_0^2$ is a copy of $\mathbb{P}^{n-1}_k$. 

(ii) Show that the natural morphism
\[ \pi : C^3_0 \to C^2_0, \]
realises $C^3_0$ as an affine bundle of dimension $n - 1$ over $C^2_0 = \mathbb{P}^{n-1}$.
(iii) Show that $\pi$ has a section, and that with this section $\pi$ becomes a vector bundle of rank $n - 1$ over $\mathbb{P}^{n-1}$. 