HWK #2, DUE WEDNESDAY 02/17

Hartshorne: Chapter II, 4.8, 4.9. **Challenge Problem** Chapter II, 4.10.

1. Let $X \subset \mathbb{P}^n_K$ be a quasi-projective toric variety. Suppose that the natural morphism $X \longrightarrow \mathbb{P}^n_K$ is a toric morphism (we give $\mathbb{P}^n_K$ the standard structure of a toric variety). Prove that there are binomials $F_1, F_2, \dots, F_p$ and monomials $G_1, G_2, \dots, G_q$, such that $x \in X$ if and only if $F_i(x) = 0$ for every $1 \leq i \leq p$ and $G_j(x) \neq 0$ for some $1 \leq j \leq q$. Now suppose that X is projective. Show that X contains the point $[1 : 1 : 1 : \cdots : 1]$ (that is, the identity of the torus) if and only if the corresponding equations can be put in the form monomial equals monomial.

2. Suppose that σ is a cone in $N_{\mathbb{R}} = \mathbb{R}^n$ and that $u_1, u_2, \dots, u_m$ are generators of the semigroup $S_\sigma \subset M$. Show that the affine toric variety $U_\sigma \subset \mathbb{A}^m_K$ is defined by monomial equations of the form

$$x_1^{a_1} x_2^{a_2} \dots x_m^{a_m} = x_1^{b_1} x_2^{b_2} \dots x_n^{b_n},$$

where

$$\sum a_i u_i = \sum b_i u_i,$$

in S_σ.

3. Prove the converse to (1). Suppose that there are binomials $F_1, F_2, \dots, F_p$ and monomials $G_1, G_2, \dots, G_q$, such that $x \in X$ if and only if $F_i(x) = 0$ for every $1 \leq i \leq p$ and $G_j(x) \neq 0$ for some $1 \leq j \leq q$. Prove that X is a quasi-projective toric variety and that the natural inclusion is a toric morphism. Give an example to show we cannot drop the normal hypothesis.