MODEL ANSWERS TO HWK #9

Let $X \subset \mathbb{P}^n$ be a closed subset, which does not contain the hyperplane H given by $X_0 = 0$. Suppose that the polynomials $Q_1, Q_2, \dots, Q_k$ have the following properties:

- they are all homogeneous of the same degree e,
- the polynomials $q_i = Q_i/X_0^d$ all have degree e,
- $q_1, q_2, \dots, q_k$ generate the ideal J of $U_0 \cap X \subset \mathbb{A}^n$.

Then the polynomials $Q_1, Q_2, \dots, Q_k$ generate the ideal I of X.
Indeed, pick $F \in I$ of degree d. As H is not contained in X we may suppose that F is not divible by X_0. If $f = F/X_0^d$ then $f \in J$. By assumption we may find $f_1, f_2, \dots, f_k$ such that $f = \sum f_i q_i$. As q_i all have the same degree, we may assume that $f_1, f_2, \dots, f_k$ all have degree $d - e$. Let $F_i = X_0^{d-e} f_i$, a homogeneous polynomial of degree $d - e$. Then $F = X_0^d f = \sum F_i Q_i$.
1. Let $I_0 = (d, 0, 0, \dots, 0)$. Then the polynomials

$$q_{I,J,K,L} = \frac{Q_{I,J,K,L}}{Z_{I_0}^2} = \frac{Z_I Z_J - Z_K Z_L}{Z_{I_0}^2},$$

all have degree 2 and so it suffices to prove that these generate the ideal of $X \cap U_0$. But we have already seen that the polynomials

$$Q_{I,J,K,L},$$

cut out X set-theoretically. Let $z_I = Z_I/Z_{I_0}$. It suffices to check that the ring

$$\frac{K[z_I]}{\langle q_{I,J,K,L} \rangle},$$

is an integral domain. But it is easy to check that the morphism given in class sets an isomorphism of this ring with the coordinate ring of $\mathbb{A}^n$.
2. As the polynomials

$$q_{ijkl} = \frac{Q_{ijkl}}{Z_{00}^2} = \frac{Z_{ij} Z_{kl} - Z_{il} Z_{jk}}{Z_{00}^2},$$

all have degree 2, it suffices to prove that these generate the ideal of $X \cap U_0$. But we have already seen that the polynomials

$$Q_{ijkl},$$

cut out X set-theoretically. Let $z_{ij} = Z_{ij}/Z_{00}$. It suffices to check that the ring

$$\frac{K[z_{ij}]}{\langle q_{ijkl}\rangle},$$

is an integral domain. But it is easy to check that the morphism given in class sets an isomorphism of this ring with the coordinate ring of $\mathbb{A}^{m+n}$.

2.14. (a) We first show that the intersection of the homogeneous prime ideals is the set of nilpotent elements of S. Indeed, the intersection of the homogeneous prime ideals certainly contains all the nilpotent elements. Suppose $s \in S$ is not nilpotent. It remains to find a homogeneous prime ideal which does not contain s. As the ideal generated by the nilpotent elements is homogeneous we may assume that s is homogeneous. Pick a maximal homogeneous ideal $\mathfrak{p}$ which does not contain s. Then $\mathfrak{p}$ is a homogeneous prime ideal which does not contain s.

Now $\operatorname{Proj} S$ is empty if and only if every homogeneous prime ideal contains S_+. So $\operatorname{Proj} S$ is empty if and only if every element of S_+ is nilpotent.

(b) Let $\mathfrak{a}$ be the homogeneous ideal generated by $\phi(S_+)$. Then $U = \operatorname{Proj} T - V(\mathfrak{a})$ and so U is open. If $g \in S$ is homogeneous then $\phi\colon S \longrightarrow T$ induces a ring homomorphism $\phi_{(g)}\colon S_{(g)} \longrightarrow T_{(\phi(g))}$. This defines a morphism $\operatorname{Spec} T_{(\phi(g))} \longrightarrow \operatorname{Spec} S_{(g)}$ whence, by composition, a morphism $\operatorname{Spec} T_{(\phi(g))} \longrightarrow \operatorname{Proj} S$. On the other hand, the sets $\operatorname{Proj} T - V(\phi(g))$ form an open cover of U. As these morphisms are clearly compatible on overlaps, this induces a morphism

$$f\colon U \longrightarrow X = \operatorname{Proj} S.$$

(c) Suppose that $\mathfrak{p}$ is a homogeneous prime ideal which contains $\phi(S_+)$. Then $\mathfrak{p}$ contains T_d, for all $d \geq d_0$. Suppose that $g \in T_d$, $d \geq 1$. Then $g^k \in T_{kd}$ and for k large enough $g^k \in \mathfrak{p}$. But then $g \in \mathfrak{p}$ and $\mathfrak{p} \supset T_+$. So $U = \operatorname{Proj} T$.

Suppose that $g \in S$ is homogeneous of degree $d \geq d_0$. Consider the ring homorphism:

$$\phi_{(g)}\colon S_{(g)} \longrightarrow T_{(\phi(g))}.$$

Let $h = \phi(g)$. Suppose that $b/h^k \in T_{(h)}$. Then $b \in T_{dk}$. Pick $a \in S_{dk}$ such that $\phi(a) = b$. Then $\phi_{(g)}(a/g^k) = b/h^k$ and so $\phi_{(g)}$ is surjective. Suppose that a/g^k maps to zero, for some $k > 0$. Then $h^l\phi(a) = 0$, in $T_{(k+l)d}$ and it follows that $g^l a = 0$ in $S_{(k+l)d}$. Thus $\phi_{(g)}$ is a ring isomorphism.

Now suppose that g is any homogeneous element of S. Then g^k is also homogeneous and if k is sufficiently large then g^k has degree at least

d_0, and $V(g) = V(g^k)$. Thus open sets of the form $\operatorname{Proj} S - V(g)$ and $\operatorname{Proj} T - V(g)$ cover $\operatorname{Proj} S$ and $\operatorname{Proj} T$, where g has degree at least d_0. It follows that f is an isomorphism.

It remains to find an example of this phenomena. Let $S = k[X,Y]/\langle X^2, XY, Y^2\rangle$ and let $T = k[X,Y]/\langle X, Y\rangle$. Then there is a natural ring homomorphism

$$\phi\colon S \longrightarrow T.$$

This map is not an isomorphism but ϕ_d is a isomorphism of vector spaces unless $d = 1$ (indeed it is the zero map between vector spaces of dimension zero, as soon as $d \geq 2$). In fact more generally take any projective variety $X \subset \mathbb{P}^n$, let $J = I(X)$ be the homogeneous ideal of X and let I be any ideal which cuts out X scheme theoretically. Let $R = k[X_0, X_1, \dots, X_n]$, $S = R/J$ and $T = R/I$.

(d) Suppose that $V \subset \mathbb{P}^n$. Then $V_i = V \cap U_i$ forms an open affine cover of V, where U_i is the standard affine open subset of $\mathbb{P}^n$. Then $t(U_i)$ forms an open cover of V. We have already seen that $t(U_i) = \operatorname{Spec} A_i$, where A_i is the coordinate ring of V_i. But $A_i = S_{(X_i)}$. It follows that there is a natural isomorphism

$$f_i'\colon t(U_i) \longrightarrow \operatorname{Proj}(S) - V(X_i),$$

and by composition we get a morphism,

$$f_i\colon t(U_i) \longrightarrow \operatorname{Proj}(S).$$

As these morphisms are compatible on overlaps, we get a morphism

$$f\colon t(V) \longrightarrow \operatorname{Proj}(S).$$

Clearly we may also define a morphism

$$g\colon \operatorname{Proj}(S) \longrightarrow t(V),$$

using the same argument. As f and g are inverse morphisms, f is an isomorphism.

3.11 (a) We first check this in the special case when $X' \longrightarrow X$ is an open immersion. In this case the image of Y' is clearly closed, the restricted morphism is a homeomorphism and surjectivity of $\mathcal{O}_{X'} \longrightarrow f_*\mathcal{O}_{Y'}$ is clear. In particular, it is easy to deduce that f is a closed immersion if and only if there is a cover by open immersion $X' \longrightarrow X$ (meaning simply that X is the union of the images) such that f' is a closed immersion, for every open set of the cover.

So to check the general case, we may assume that $X = \operatorname{Spec} A$ is affine. Let $V \subset Y$ be an open affine subset of Y. We may find an open subset $U \subset X$ such that $f^{-1}(U) = V$. Then we may find a regular function f on X (or better $f \in A$) such that $U_f \subset U$. Then $f^{-1}(U_f)$ is an open affine subset of V. Since U_f cover U, we may assume that X and

$Y = \operatorname{Spec} B$ are both affine. In this case B is a quotient of A. Finally we may assume that $X' = \operatorname{Spec} A'$ is affine. Since $B' = B \underset{A}{\otimes} A'$ is a quotient of A', f' is indeed a closed immersion.

(b) Pick an open affine cover $\{Y_\alpha\}$ of Y. Then there is an open subset X_α of X such that $Y_\alpha = Y \cap X_\alpha$. We may find f_i such that for every α there is an index i such that $U_{f_i} \subset X_\alpha$. Then $U_{f_i} \cap Y$ is an open affine subset of Y, as it is equal to the locus where the regular function $f|Y_\alpha$ is not zero on the affine scheme Y_α. By compactness we may assume there are only finitely many $f_1, f_2, \dots, f_r$. $f_1, f_2, \dots, f_r$ generate the unit ideal as the sets U_{f_i} are an open affine cover of X. By (2.17.b) Y is affine. Now apply (2.18.d).

(c) We want to give a morphism of schemes $Y \longrightarrow Y'$. The map on topological spaces is simply the identity. Pick an open affine cover of X. By part (b) this induces an open affine cover of Y and Y'. On this affine cover if Y and Y' are given by ideals $\mathfrak{a}$ and $\mathfrak{a}'$ in the ring A, then $\mathfrak{a}$ is the radical of $\mathfrak{a}'$. In particular there is a natural inclusion $\mathfrak{a} \subset \mathfrak{a}'$ and so a natural surjection $A/\mathfrak{a}' \longrightarrow A/\mathfrak{a}$ which factors $A \longrightarrow A/\mathfrak{a}'$ and $A \longrightarrow A/\mathfrak{a}$. This gives us a commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & Y' \arrow[d] \\ & X. \end{tikzcd}$$

These maps automatically glue, by naturality.

(d) We first suppose that $X = \operatorname{Spec} A$ is affine. In this case there is a homomorphism of rings,

$$A \longrightarrow H^0(Z, \mathcal{O}_Z).$$

Let $\mathfrak{p}$ be the kernel and let B be the quotient, so that there is a ring commutative diagram,

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & B \arrow[d] \\ & H^0(Z, \mathcal{O}_Z). \end{tikzcd}$$

Let $Y = \operatorname{Spec} B$. Then, Y is a closed subscheme of X and there is a commutative diagram

$$\begin{tikzcd} X & Y \arrow[l] \\ & Z. \arrow[ul] \arrow[u] \end{tikzcd}$$

Now suppose that there is another commutative diagram,

$$\begin{tikzcd} X & Y' \arrow[l] \\ & Z. \arrow[ul] \arrow[u] \end{tikzcd}$$

Then there is an induced map of rings,

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & H^0(Y', \mathcal{O}_{Y'}). \arrow[d] \\ & H^0(Z, \mathcal{O}_Z). \end{tikzcd}$$

By the universal property of the quotient, there is an induced ring homomorphism,

$$H^0(Y', \mathcal{O}_{Y'}) \longrightarrow B,$$

and this gives rise to a morhism of schemes $Y \longrightarrow Y'$.
Now suppose that X is arbitrary. Pick an open affine cover $\{U_i\}$ of X, such that U_{ij} is affine. Let V_i be the inverse image of U_i in X_i. Let $g_i \colon Y_i \longrightarrow X$ be the affine scheme constructed above. Let $Y_{ij} = g_i^{-1}(U_j)$ be the inverse image of U_j. Then Y_{ij} and Y_{ji} satisfy the same universal property and so there are induced isomorphisms ϕ_{ij} which satisfy the cocycle condition. Glueing together the Y_i, this defines Y. Y is a closed subscheme of X and it clearly satisfies the given universal property.
The last property is clear, since both Y and the reduced induced subscheme enjoy the same universal property.
3.12. (a) $\phi(S_+) = \phi(T_+)$, as ϕ is surjective, and so $U = \operatorname{Proj} T$. Now suppose that $g \in T$ is homogeneous. If $h = \phi(g) \in S$ then

$$\phi_{(g)} \colon S_{(h)} \longrightarrow T_{(g)},$$

is surjective. Therefore

$$f_{(g)} \colon \operatorname{Proj} T - V(h) = \operatorname{Spec} T_{(h)} \longrightarrow \operatorname{Proj} S - V(g) = \operatorname{Spec} S_{(g)},$$

is a closed immersion. As open sets of the form $\operatorname{Proj} S - V(g)$ cover $\operatorname{Proj} S$ it follows that f is a closed immersion.
(b) We have surjective ring homomorphisms $S \longrightarrow S/I'$, $S \longrightarrow S/I$ and $S/I' \longrightarrow S/I$. This gives rise to closed immersions $i\colon \operatorname{Proj} S/I' \longrightarrow \operatorname{Proj} S$, $j\colon \operatorname{Proj} S/I \longrightarrow \operatorname{Proj} S$ and $k\colon \operatorname{Proj} S/I \longrightarrow \operatorname{Proj} S/I'$, such that $j = i \circ k$. k is an isomorphism by (2.14.c) and so i and j are equivalent closed immersions. By (2.14.d) there are plenty of examples of this phenomena.
3.13 (a) Let $f \colon X \longrightarrow Y$ be a closed immersion. Suppose that $i \colon U \longrightarrow Y$ is an open immersion, where U is affine. By (3.11.a) the map

$g \colon V \longrightarrow U$ obtained by pulling back the morphism f along the morphism i is a closed immersion. As U is affine, (3.11.b) implies that V is affine as well, and the map i is induced by a quotient ring homomorphism,

$$A \longrightarrow B = A/\mathfrak{a}.$$

B is clearly a finitely generated A-algebra and so f is of finite type.
(b) Let $f \colon X \longrightarrow Y$ be an open immersion. Let $U \subset X$ be an affine open subset of X. Then $f(U)$ is an open affine subset of Y which is isomorphic to U. It follows that f is locally of finite type and as f is quasi-compact, it is of finite type.
(c) Let $f \colon X \longrightarrow Y$ and $g \colon Y \longrightarrow Z$ be two morphisms of finite type and let $h \colon X \longrightarrow Z$ be the composition. Pick an open affine subset $W = \operatorname{Spec} C$ of Z. By (3.3.b) we may find a finite open affine cover $V_i = \operatorname{Spec} B_i$ of $g^{-1}(W)$ such that B_i is a finitely generated C-algebra. For each V_i, we may find a finite open affine cover $U_{ij} = \operatorname{Spec} A_{ij}$ of $f^{-1}(V_i)$, such that C_{ij} is a finitely generated B_i-algebra.
But then $U_{ij} = \operatorname{Spec} A_{ij}$ is a finite open affine cover of $h^{-1}(W)$ where A_{ij} is a finitely generated C-algebra. Therefore h is of finite type.
(d) Let $f \colon X \longrightarrow Y$ be a morphism of finite type and let $Y' \longrightarrow Y$ be a morphism. Let $f' \colon X' \longrightarrow Y'$ be the induced morphism, where X' is the fibre product of X and Y' over Y. We want to prove that f' is of finite type. Let $V = \operatorname{Spec} B$ be an open subset of Y. Then there is a finite open affine cover $U_i = \operatorname{Spec} A_i$ of $f^{-1}(V)$, where A_i is a finitely generated B-algebra.
(e) By part (d) $X \underset{S}{\times} Y \longrightarrow Y$ is of finite type. But then the morphism $X \underset{S}{\times} Y \longrightarrow S$ is of finite type, as it is a composition of morphisms of finite type.
(f) Let $W = \operatorname{Spec} C$ be an affine open subset of Z. By assumption $(g \circ f)^{-1}(W)$ can be covered by affine open subsets $U = \operatorname{Spec} A$ of X, where A is a finitely generated C-algebra. Pick an affine open subset $V = \operatorname{Spec} B$ of $g^{-1}(W)$. Then we can cover $f^{-1}(V) \cap U$ with affine open subsets of the form $\operatorname{Spec} A_h$, where h is a regular function on U. As A_h is a finitely generated C-algebra it is a finitely generated B-algebra. But then f is locally of finite type and as f is compact, it is of finite type.
(g) Let $V = \operatorname{Spec} B$ be an affine subset of Y. Then $f^{-1}(V)$ is a finite union of affine sets of the form $U = \operatorname{Spec} A$, where A is a finitely generated B-algebra. As B is Noetherian, A is Noetherian and so X is Noetherian.
3.15 We first make some general observations that apply to both parts (a) and (b).

Suppose that X is of finite type over a field k. Then X has a finite cover $U_i = \operatorname{Spec} A_i$ by open affines, where A_i is a finitely generated k-algebra. If U_i and U_j don't intersect then $U_i \cup U_j = \operatorname{Spec} A_i \oplus A_j$ is affine. So we may assume that $U_i \cap U_j$ is non-empty. But then X is irreducible or reduced if and only if U_i is irreducible or reduced, for all i.

If K/k is any field extension, then $Y = X \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} K$ is covered by open affines of the form $V_i = \operatorname{Spec} B_i = \operatorname{Spec} A_i \underset{k}{\otimes} K$. As $U_i \cap U_j$ is non-empty, so is $V_i \cap V_j$. Thus Y is irreducible or reduced if and only if V_i is irreducible or reduced for all i.

So we might as well assume that $X = \operatorname{Spec} A$ is affine. If X is non-reduced or reducible then so is $Y = \operatorname{Spec} B$. So we may assume that X is integral and Y is not integral, and we may assume that K contains the algebraic closure of k. As A is a finitely generated k-algebra, it is a quotient of a polynomial ring $k[x_1, x_2, \dots, x_n]$ by an ideal $\mathfrak{p}$, which is prime, as A is an integral domain. By assumption there are $f(x)$ and $g(x)$ in $K[x_1, x_2, \dots, x_n]$, neither of which belong to $\mathfrak{q}$, the ideal generated by $\mathfrak{p}$, whose product does belong to $\mathfrak{q}$. Suppose that f has degree d and g has degree e.

Now suppose that $W \subset \mathbb{A}^m_k$ is a quasi-projective scheme, defined by the vanishing of $F_1, F_2, \dots, F_r$ and the non-vansihing of $G_1, G_2, \dots, G_s$. We now introduce some convenient notation (which happily is also quite standard). Given a ring R/k, let $W(R)$ denote the set of all maps $\operatorname{Spec} R \longrightarrow W$ (any such is called an R-valued point of W; Yoneda's Lemma pretty much says that we can recover W from the data of all of these sets, for all such rings R). Note that $W(K)$ is simply the set of points $(a_1, a_2, \dots, a_m)$ with coordinates in K which satisfy the polynomials $F_1, F_2, \dots, F_r$ but not the polynomials $G_1, G_2, \dots, G_s$.

Suppose that $W(K)$ is non-empty, for some field extension K/k. I claim that this implies that $W(\bar{k})$ is non-empty. The proof proceeds by induction on m. The case $m = 1$ is almost the definition of algebraically closed. Pick a point p of $\mathbb{A}^m_k(k)$. We may assume that $p \notin W(k)$ (else there is nothing to prove). The image W' of W is a constructible subset of $\mathbb{A}^{m-1}$. By induction on m, $W'(\bar{k})$ is non-empty. Pick a point $q \in W'(\bar{k})$ and let l be the line connecting p to q. Then l is defined by linear polynomials with coefficients in $\bar{k}$. $(l \cap W)(K)$ has a point and so $W(\bar{k})$ has a point (which lies on the line l).

Now suppose that $W(\bar{k})$ is non-empty. Pick a point $(a_1, a_2, \dots, a_m) \in W(\bar{k})$. Let k' be the field generated by $a_1, a_2, \dots, a_m$. Then k'/k is a finitely generated field extension which is finite as k'/k is algebraic.

(a) Suppose that $B = A \underset{k}{\otimes} K$ is not an integral domain even though it does not have any nilpotents. It follows that there are polynomials f and $g \in K[x_1, x_2, \dots, x_n]$ whose images in B are non-zero, but whose product is zero. By assumption there are also points p and $q \in X(K)$ such that f does not vanish at x and g does not vanish at y. We may assume that p and q belong to $X(\bar{k})$. Suppose that the degree of f is d and the degree of g is e.

The space of all polynomials of degree d with coefficients in k is naturally identified with an affine space $P_d \simeq \mathbb{A}_k$ (of dimension a function of d). Now multiplication of polynomials defines a function

$$\mathbb{A}^m_k = P_d \times P_e \longrightarrow P_{d+e},$$

for some appropriate m, which is easily seen to be a morphism. The locus W of polynomials f and g which do not lie in the affine space corresponding to $\mathfrak{p}$ but whose product does lie in the third, such that f does not vanish at x and g does not vanish at y, is defined by the vanishing of some polynomials $F_1, F_2, \dots, F_r$ and the non-vanishing of $G_1, G_2, \dots, G_s$. By assumption $W(K)$ is non-empty. But then $W(\bar{k})$ is non-empty.

In fact we know that $W(k')$ is non-empty, for some finite extension k'/k and we may assume that x and y belong to $X(k')$. By induction on the order of the extension, we may assume that $k' = k_1(\alpha)$, where $\alpha^p \in k_1$ and p is the characteristic of k. Pick a point $(f, g) \in W(k')$, where f and g are polynomials with coefficients in k'. Then f^p and g^p have coefficients in k_1. Let $x' = ({x'}_1, {x'}_2, \dots, {x'}_m) \in W(k_1)$, where $x'_i = x^p_i$. Then

$$f^p(x') = (f(x))^p \neq 0.$$

Define y' similiarly. Then $X_1 = X \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} k_1$ is reducible and this completes the induction.

(b) If B has nilpotents even though A is integral then we may find $f \in K[x_1, x_2, \dots, x_n]$ such that $f^e = 0$ in B. Suppose that f has degree d. Consider the morphism

$$P_d \longrightarrow P_{de},$$

induced by raising to the power e. Let W be the locus consistng of polynomials f which don't belong $\mathfrak{p}$ but whose image does. $W(K)$ is non-empty by assumption. But then $W(\bar{k})$ is non-empty.

As before, this implies that $W(k')$ is non-empty, for some finite extension k'/k. We may split this extension into two parts, k'/k_1 and k_1/k, where $k_1 \subset k_p$ is a purely inseparable extension of k, and k'/k_1 is separable. Passing to the normal closure of k', we may assume that k'/k_1 is

Galois. Let $f_1 \in W(k')$ and let $f_1, f_2, \ldots, f_k$ be the Galois conjugates of f_1. Then their product $f \in W(k_1)$. By part (a), $X_1 = X \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} k_1$ is irreducible, so that f is non-zero. But then f is nilpotent.

(c) Consider $\operatorname{Spec} k(t)[x]/\langle x^6 - t\rangle$, where $k = \mathbb{F}_2$. As $x^6 - t \in k(t)[x]$ is irreducible, this scheme is integral. But if we replace t by t^6 (equivalently, we adjoin a root of $x^6 - t$) then $x^3 - t^3$ is a non-zero global section which squares to zero and $x^3 - t^3$ factors non-trivially, so that $\operatorname{Spec} k(t)[x]/\langle x^6 - t^6\rangle$ is neither reduced nor irreducible.