MODEL ANSWERS TO HWK #8

2.16. (a) Suppose that $x \in U$. As U is open,

$$\mathcal{O}_{U,x} \simeq \mathcal{O}_{X,x},$$

and the rest is clear.

(b) As X is compact there is an open cover $\{U_1, U_2, \dots, U_k\}$ of X by finitely many affines. By our answer to part (a), $X_f \cap U_i = U_{f_i}$, where f_i is the restriction of f to U_i. As a is zero on X_f, its restriction a_i to U_{f_i} is zero. As $U_i = \operatorname{Spec} A_i$ is affine, it follows that $U_{f_i} = \operatorname{Spec}(A_i)_{f_i}$. In particular $f_i^{n_i} a_i = 0$, for some $n_i \in \mathbb{N}$. As we have a finite cover, we may assume that $n = n_i$ is independent of i. We may also assume $n > 0$. Since the restriction of $f^n a \in \Gamma(X, \mathcal{O}_X)$ to each set U_i of the open cover $\{U_1, U_2, \dots, U_k\}$ is zero, it follows that $f^n a$ is zero.

(c) Let b_i be the restriction of b to U_i and let f_i be the restriction of f to U_i. As U_i is affine and $X_f \cap U_i = U_{f_i}$ by part (a), we may lift $f_i^{n_i} b_i$ to c_i on U_i. Now $c_i - c_j$ restricts to zero on $U_{ij} \cap X_f$. As we are assuming that U_{ij} is compact, it follows that $c_i - c_j$ resricts to zero on the whole of U_{ij}, by our answer to part (b). But then there is a section c on the whole of X which restricts to c_i on U_i. The axioms for a sheaf also imply that c is a lift of $f^n b$.

(d) Note first that X is compact as it has a finite cover by open affines, which are always compact.

Consider the natural restriction map

$$A = \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(X_f, \mathcal{O}_{X_f}).$$

As f is sent to a unit, there is a natural map

$$A_f \longrightarrow \Gamma(X_f, \mathcal{O}_{X_f}).$$

The answer to part (b) proves that this map is injective and the answer to part (c) that it is surjective. Hence this map is an isomorphism.

2.17 (a) The map on topological spaces is surely a homeomorphism under these circumstances. It suffices, then, to check that the map on structure sheaves is an isomorphism. As this may be checked on stalks, the result follows.

(b) If X is affine, just take $r = f = 1$.

Otherwise suppose that we have $f_1, f_2, \dots, f_r$ such that U_{f_i} is affine, where $f_1, f_2, \dots, f_r$ generate the unit ideal. Let $Y = \operatorname{Spec} A$. By (2.4) there is a morphism

$$f \colon X \longrightarrow Y,$$

induced by the identity map $A \longrightarrow A$. Let V_{f_i} be the open affine subset of Y where f_i is not zero. Then $f^{-1}(V_{f_i}) = U_{f_i}$ and both sets are affine. By our answer to (2.16.d), they are both isomorphic to $\operatorname{Spec} A_{f_i}$ and the induced map on A_{f_i} is the identity. So the morphism f is certainly an isomorphism over the open subset V_{f_i}. But since $f_1, f_2, \dots, f_r$ generate the unit ideal, these sets cover X and we are done by part (a).

Before we prove the next exercises, we recall a result that was proved implicitly in the lectures. Suppose that X is a scheme and that $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ are two affine schemes. Then $U \cap V$ be covered by finitely many affine schemes which are simultaneously isomorphic to U_g and V_h, where $g \in A$ and $h \in B$.

3.1 It suffices to prove that $f^{-1}(V)$ is covered by open affines $U = \operatorname{Spec} A$ such that A is a finitely generated B-algebra. By the observation above we may assume that $Y = V = \operatorname{Spec} B$ and that we can cover Y by finitely many open affine subsets $V_i = U_{f_i}$, where $f^{-1}(V_i)$ can be covered by open affines which are the spectra of finitely generated B_{h_i}-algebras. For each i, pick $U_i = \operatorname{Spec} A_i$ lying over V_i where A_i is a finitely generated B_i-algebra.

Let U be the union of $U_1, U_2, \dots, U_k$. As sets of this form cover $f^{-1}(Y)$ it suffices to prove that U is an open affine which is the spectrum of a finitely generated B-algebra. It is clear that U is open.

Let g_i be the image of f_i in $A = \Gamma(U, \mathcal{O}_U)$. Then U_i is the locus where g_i is not equal to zero. $g_1, g_2, \dots, g_k$ generate the unit ideal of A, as $f_1, f_2, \dots, f_k$ generate the unit ideal of B. It follows by (2.17), that U is affine and it suffices to prove that A is a finitely generated B-algebra.

So now we are reduced to the following problem in algebra. Let B be an A-algebra, and let $f_1, f_2, \dots, f_k$ generate the unit ideal. Suppose that g_i is the image of f_i and suppose that $A_i = A_{g_i}$ is a finitely generated $B_i = B_{f_i}$-algebra. Then A is a finitely generated B-algebra.

We now prove this result in commutative algebra. To this end, pick generators $c_{i1}, c_{i2}, \dots, c_{il_i}$ of A_i over B_i. Then each c_{ij} has the form a_{ij}/g_i^n, where we may assume that n is constant, as we have only finitely many indices. I claim that a_{ij}, for every i and j, generates A over B. Pick $a \in A$. Then if $\phi_i \colon A \longrightarrow A_i$ is the natural map, we have

$$\phi_i(a) = p(c_{ij}),$$

for some for some polynomial p, with coefficients in B_i. Clearing denominators, we then have

$$g_i^N a = q(a_{ij}),$$

for some polynomial q, with coefficients in A_i. We may write

$$\sum h_i g_i^N = 1,$$

for some h_i. But then

$$\begin{aligned} a &= \sum h_i g_i^N a, \\ &= \sum h_i \left(\sum_j q(a_{ij}) \right), \end{aligned}$$

as required.

3.2 The key observation is that a scheme is compact iff it is the finite union of affine subschemes. Indeed, if X is a scheme, then it is union of open affine subschemes, and if X is compact, then finitely many cover. Conversely, any affine scheme is compact, and the finite union of compact sets is always compact.

So now suppose that $f\colon X \longrightarrow Y$ is a compact morphism, and let V be an affine subset. Using the argument just before (3.1) we may assume that $Y = V$. Let V_i be an open affine cover of Y such that $f^{-1}(V_i)$ is compact. As Y is affine we may assume that this cover is finite. But then $f^{-1}(Y)$ is compact, as it is a finite union of compact subsets.

3.3 (a) Clear, from the first paragraph of 3.2.

(b) Simply apply 3.1 and 3.2.

(c) By now standard tricks, we can reduce this problem to showing that if a B-algebra A contains elements $f_1, f_2, \dots, f_k$ which generate the unit ideal and A_{f_i} is a finitely generated B-algebra, then so is A. But this is easily implied by part of the proof of 3.1.

3.4 Follows almost exactly the same proof as 3.1, and 3.3 (c). We are reduced to proving that if A is a B-algebra and $f_1, f_2, \dots, f_k$ are elements of B which generate the unit ideal, such that $A_i = A_{g_i}$ is a finitely generated $B_i = B_{f_i}$-module, where g_i is the image of f_i, then A is a finitely generated B-module.

Repeating the argument given in (3.1), we are given $a_{ij} \in A$ whose images c_{ij} under ϕ_i generate A_i as a B_i-module. As before this implies that

$$g_i^N a = \sum_j b_{ij} a_{ij},$$

for some $b_{ij} \in B$. It is then easy to see that we may write a as a linear combination of a_{ij}, so that the a_{ij} generate A as a B-module.

3.6 Let $U = \operatorname{Spec} A$ be any open affine subscheme. Then $\xi \in U$ and so ξ corresponds to a prime ideal of A, which is the the zero ideal. But then

$$\mathcal{O}_{X,\xi} \simeq A_{\langle 0 \rangle} = K,$$

where K is the field of fractions of A.

3.8 We have to check the patching condition. Suppose that $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ are two affine open subschemes of X. Let $\tilde{U} = \operatorname{Spec} \tilde{A}$ and $\tilde{V} = \operatorname{Spec} \tilde{B}$. We have to exhibit a canonical isomorphism

$$\phi \colon U' \longrightarrow V',$$

where U' is the inverse image of $U \cap V$ in $\tilde{U}$ and V' is the inverse image of $U \cap V$ in $\tilde{V}$.

Since it suffices to construct a canonical morphism on an open cover, we may assume that U and V are open affines of a common affine scheme $W = \operatorname{Spec} C$ and that $A = C_f$ and $B = C_g$, where f and g belong to C. It suffices to check that if $\tilde{A}$ is the integral closure of A, then $\tilde{A}_f$ is the integral closure of A_f. It is clear that any element of $\tilde{A}_f$ is integral over A_f. Indeed if $a/f^k \in \tilde{A}_f$, where $a \in \tilde{A}$ satisfies the monic polynomial

$$x^n + a_{n-1}x^n + \cdots + a_0 \in A[x],$$

then a/f^k satisfies the monic polynomial

$$x^n + b_{n-1}x^{n-1} + \cdots + b_0 \in B[x],$$

where $b_i = a_i/f^{k(n-i)}$. On the other hand if u belong to the integral closure of A_f, then u is a root of a monic polynomial

$$x^n + b_{n-1}x^{n-1} + \cdots + b_0,$$

where each $b_i \in A_f$. Clearing denominators, it follows that $a = f^l u \in \tilde{A}$, for an appropriate power of f. Hence $\tilde{A}_f$ is the integral closure of A_f.

Thus one can glue the schemes $\tilde{U}$ together to get a scheme $\tilde{X}$. The inclusion $A \longrightarrow \tilde{A}$ induces a morphism of schemes $\tilde{U} \longrightarrow U$, whence a morphism of schemes $\tilde{U} \longrightarrow X$. Arguing as before, these morphisms agree on overlaps. It follows that there is an induced morphism $\tilde{X} \longrightarrow X$.

Now suppose that there is a dominant morphism of schemes $Z \longrightarrow X$, where Z is normal. This induces a dominant morphism $Z_U \longrightarrow U$, where U is an open affine subscheme and Z_U is the inverse image of U Thus it suffices to prove the universal property of X in the case when X is affine. Covering Z by open affines, it suffices to prove this result when Z is affine. Using the equivalence of categories, we are reduced to proving that if $A \longrightarrow \tilde{A}$ is the inclusion of A inside its integral closure, and $A \longrightarrow B$ is a ring homomorphism, where B is integrally closed, then there is a ring homomorphism $\tilde{A} \longrightarrow B$. Clearly there is such a morphism into the field of fractions L of B. On the other hand, any element of the image is obviously integral over the image of A, and so integral over B. But then the image of $\tilde{A}$ lies in B, as B is integrally closed.

Suppose that X is of finite type. Clearly we may assume that $X = \operatorname{Spec} A$ is affine. We are reducing to showing that the integral closure $\tilde{A}$ of a finitely generated k-algebra A, is a finitely generated A-module. But this was proved in the lectures.

3.9 (a)

$$\mathbb{A}^2_k = \operatorname{Spec} k[x,y] = \operatorname{Spec}(k[x] \underset{k}{\otimes} k[y]) = \mathbb{A}^1_k \underset{k}{\times} \mathbb{A}^1_k.$$

The points of $\mathbb{A}^1_k$ consist of the maximal ideals m_a and the generic point ξ. The points of the product of sets are then ordered pairs (m_a, m_b), with closure $\{(m_a, m_b)\}$, (m_a, ξ), with closure

$$\{\, (m_a, m_b) \,|\, b \in k \,\} \cup \{(m_a, \xi)\},$$

(ξ, m_b) with closure

$$\{\, (m_a, m_b) \,|\, a \in k \,\} \cup \{(\xi, m_b)\},$$

and (ξ, ξ), whose closure is the whole space. Let $\eta = (xy - 1)$. Then η is a prime ideal, whose closure is the set

$$\{\, (m_a, m_b) \,|\, ab = 1 \,\} \cup \{\eta\}.$$

Thus η is not a point of the product of the two sets.

(b) We have

$$X = \operatorname{Spec} k(s) \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} k(t) = \operatorname{Spec}(k(s) \underset{k}{\otimes} k(t)).$$

Note however that

$$k(s) \underset{k}{\otimes} k(t) \neq k(s,t).$$

The LHS is the localisation of the polynomial ring $k[s,t]$ at the multiplicative set S generated by the irreducible polynomials in s and the irreducible polynomials in t. A typical element of the LHS is of the form

$$\frac{f(s,t)}{g(s)h(t)},$$

where $f(s,t)$ is a polynomial in s and t, $g(s)$ is a polynomial in s and $h(t)$ is a polynomial in t.

Let us try to interpret this geometrically. For ease of notation and to make it clearer what is going on, let's suppose that k is algebraically closed. This assumption makes no material difference to the answer. Since we are working with a localisation of $k[s,t]$ we expect to obtain X from $\mathbb{A}^2_k$ by throwing away some points. Let us first consider the maximal ideals. These are of the form $\langle s-a, t-b\rangle$. Since we invert every polynomial in s and t, these ideals become the unit ideal in $k[s,t]_S$. Furthermore any ideal of the form $\langle s-a\rangle$ or $\langle t-b\rangle$ becomes the unit ideal as well. So we obtain X from $\mathbb{A}^2_k$ by throwing away all of the maximal ideals, that is all the closed points, and all ideals of the form $\langle s-a\rangle$ or $\langle t-b\rangle$, that is all horizontal and vertical lines. What remains is the generic point and points corresponding to irreducible curves in $\mathbb{A}^2_k$, other than horizontal or vertical lines. In fact X is in some respects very much like a curve (one generic point and infinitely many closed points).

Note one key point about this example. $k(t)$ is not a finitely generated k-algebra, so that $\operatorname{Spec} k(t) \longrightarrow \operatorname{Spec} k$ is not a morphism of finite type.