MODEL ANSWERS TO HWK #7

2.1 By the universal property of the localisation there is a ring homomorphism

$$\phi\colon A \longrightarrow A_f.$$

Since we have an equivalence of categories between the category of rings and affine schemes, this induces a morphism of schemes

$$(g, g^{\#})\colon \operatorname{Spec} A_f \longrightarrow \operatorname{Spec} A = (X, \mathcal{O}_X).$$

Given $\mathfrak{q} \triangleleft A_f$, a prime ideal of A_f, $g(\mathfrak{q}) = \phi^{-1}(\mathfrak{q})$ is a prime ideal of A which does not contain f. Thus we get an induced morphism

$$(g, g^{\#})\colon \operatorname{Spec} A_f \longrightarrow (U_f, \mathcal{O}_{U_f} = \mathcal{O}_X|_{U_f}),$$

and it suffices to prove that this morphism is an isomorphism.
We first show that g is a homeomorphism. Now a prime ideal of $\mathfrak{q} \triangleleft A_f$ gives rise to a surjective ring homomorphism $A_f \longrightarrow B$, where $B = A/\mathfrak{q}$ is an integral domain. Composing, we get a surjective ring homomorphism $A \longrightarrow B$, and the kernel is a prime ideal $\mathfrak{p} = g(\mathfrak{q}) = \mathfrak{q} \cap A$, which does not contain f. Conversely, a prime ideal of $\mathfrak{p} \triangleleft A$ not containing f gives rise to a surjective ring homomorphism $A \longrightarrow B$, where B is an integral domain and the image f' of f is not zero. Composing with the localisation map $B \longrightarrow B_{f'}$ we get a ring homomorphism with the same kernel, and the image of f is invertible. This gives us a surjective ring homomorphism $A_f \longrightarrow B_{f'}$ by the universal property of the localisation, and the kernel $\mathfrak{q} = \mathfrak{p}A_f$ is a prime ideal of A_f. It follows that g is a bijection.
On the other hand

$$\begin{aligned}
\mathfrak{p} \in V(\mathfrak{a}) &\Leftrightarrow \mathfrak{a} \subset \mathfrak{p} \\
&\Leftrightarrow \mathfrak{a}A_f \subset \mathfrak{p}A_f \\
&\Leftrightarrow \mathfrak{p}A_f \in V(\mathfrak{a}A_f).
\end{aligned}$$

Thus g is a homeomorphism.
To see that $g^{\#}$ is an isomorphism, it suffices to check that it is an isomorphism stalk by stalk. If $\mathfrak{p} \triangleleft A$ is any prime ideal and $a \notin \mathfrak{p}$ then the image of a in $(A_f)_{\mathfrak{q}}$ is a unit, where $\mathfrak{q} = g^{-1}(\mathfrak{p})$. It follows that there is a natural ring homomorphism

$$g^{\#}_{\mathfrak{p}}\colon \mathcal{O}_{U_f,\mathfrak{p}} = \mathcal{O}_{X,\mathfrak{p}} \simeq A_{\mathfrak{p}} \longrightarrow (A_f)_{\mathfrak{q}} \simeq \mathcal{O}_{\operatorname{Spec} A_f,\mathfrak{q}},$$

To check that this ring homomorphism is an isomorphism, it suffices to check that $(A_f)_{\mathfrak{q}}$ satisfies the same universal property as $A_{\mathfrak{q}}$. Composing with ϕ, we get a ring homomorphism $A \longrightarrow (A_f)_{\mathfrak{q}}$. Every element not in $\mathfrak{p}$ is sent to a unit and it is strightforward to check that $(A_f)_{\mathfrak{q}}$ satisfies the universal property of the localisation.
2.2 The pair $(U, \mathcal{O}_U = \mathcal{O}_X|_U)$ is surely a locally ringed space, as the stalks of $\mathcal{O}_X$ and $\mathcal{O}_U$ are the same. It suffices to prove that locally this pair is isomorphic to an affine scheme. To this end, we may assume that $X = \operatorname{Spec} A$. Since open sets of the form U_f, $f \in A$ form a base for the topology, and these open sets are affine by (2.1), it suffices to observe that

$$(U_f, \mathcal{O}_X|_{U_f}) \simeq (U_f, \mathcal{O}_U|_{U_f}),$$

for any open set $U_f \subset U$.
2.3 (a) Suppose that the stalk $\mathcal{O}_{X,p}$ contains a non-zero element $f = (g, U)$ which is nilpotent. Since $f^n = 0$, possibly making U smaller, we may assume that $g^n = 0$, but $g \neq 0$. But then $\mathcal{O}_X(U)$ contains a nilpotent element.
Conversely, if $g \in \mathcal{O}_X(U)$ is nilpotent then pick a point $p \in U$ such that $g \neq 0 \in \mathcal{O}_{X,p}$. Then $f = (g, U) \in \mathcal{O}_{X,p}$ is also nilpotent.
(b) Clearly the pair $(X, \mathcal{O}_{X_{\text{red}}})$ is a locally ringed space, and so it suffices to prove that locally it is isomorphic to an affine scheme. To this end, we may assume that $X = \operatorname{Spec} A$ is affine. Note that there is a surjective ring homomorphism,

$$\phi\colon A \longrightarrow B,$$

where B is the quotient of A by the intersection of all the prime ideals, which is nothing but the set of all nilpotent elements of A. Since we have an equivalence of categories, this induces a morphism

$$(h, h^{\#})\colon \operatorname{Spec} B \longrightarrow X = \operatorname{Spec} A.$$

This induces a morphism of sheaves, between the structure sheaf of $\operatorname{Spec} B$ and $\mathcal{O}_{X_{\text{red}}}$, which is an isomorphism, since it is an isomorphism stalk by stalk.
(c) It suffices to prove this locally on X and Y. Thus we may assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are both affine schemes. There is an induced ring homomorphism $\phi\colon A \longrightarrow B$. By assumption B contains no nilpotents, thus the kernel of ϕ contains the nilpotent elements $I \triangleleft A$, and there is a natural ring homomorphism $C \longrightarrow B$, where $C = A/I$. By (b) $Y_{\text{red}} = \operatorname{Spec} C$ and it is clear that the induced morphism $X \longrightarrow Y_{\text{red}}$ has the given universal property.

2.4 Let $V = \operatorname{Spec} A$. We first check that α is injective. Since we can check this locally, we may assume that $X = \operatorname{Spec} B$ is affine, in which case we have already shown in lectures that α is bijective.
Now suppose that we are given a ring homomorphism $\phi\colon A \longrightarrow B = \Gamma(X, \mathcal{O}_X)$. Let $U_i = \operatorname{Spec} B_i$ be an open affine cover of X. Then there are natural ring homomorphisms $B \longrightarrow B_i$, given by restriction of sections. Composing we get ring homomorphisms $A \longrightarrow B_i$ and so get morphisms of schemes $U_i \longrightarrow V$. To show that we get a morphism $X \longrightarrow V$ it suffices to show that we get the same morphism on U_{ij}. But the two ways to get morphisms $U_{ij} \longrightarrow X$ both induce the same ring homomorphism $A \longrightarrow B_{ij}$, where $B_{ij} = \Gamma(U_{ij}, \mathcal{O}_X)$, since both homomorphisms are the composition of ϕ and the homomorphism $B \longrightarrow B_{ij}$ given by restriction. It follows that the two morphisms are the same, by what we already proved. Hence α is surjective.
2.5 The points of $\operatorname{Spec} \mathbb{Z}$ are the ideals generated by the prime numbers, which are closed points, together with the zero ideal, which is the generic point. The proper closed sets correspond to finite unions of prime numbers.
Since $\mathbb{Z}$ is an initial object in the category of rings $\operatorname{Spec} \mathbb{Z}$ is a terminal object in the category of affine schemes. Since every scheme is locally an affine scheme, it follows that $\operatorname{Spec} \mathbb{Z}$ is a terminal object in the category of schemes.
2.7 Since K is a field, it has a unique prime ideal, and so $\operatorname{Spec} K$ certainly has only one point, and the structure sheaf is represented by K itself. To give a morphism of $\operatorname{Spec} K$ to X, we certainly have to pick out a point $x \in X$. But then, by definition of a scheme, there is an induced morphism of local rings,

$$\mathcal{O}_{X,x} \longrightarrow K.$$

But this is equivalent to a ring homomorphism, which sends the maximal ideal m_x to zero, which in turn is equivalent to giving an inclusion of the residue field of x into K.
2.9 Let Z be an irreducible closed subset. Pick an affine open subset U of X which intersects Z. Then $V = U \cap Z$ is a dense open affine subset of Z. Now $V = \operatorname{Spec} B$, for some ring B. The nilpotent elements of B form a prime ideal. Let $\xi \in V$ be the corresponding point. Then $\{\xi\}$ is dense in V so that it is dense in Z.
Now suppose that both ξ' is dense in Z. Then $\xi' \in U$ and so $\xi' = \xi$, the ideal of all nilpotent elements.
2.11 The maximal ideals in $\mathbb{F}_p[x]$ are of the form $\langle f \rangle$, where f is irreducible. The only other ideal is the zero ideal. The proper closed subsets are finite unions of ideals of the first kind. The residue field is

a finite extension of $\mathbb{F}_p$ in the first case and otherwise it is $\mathbb{F}_p(x)$. There is one point with this residue field. If $q = p^r$ then the number of points with this residue field is the number of monic irreducible polynomials of degree r. Now the elements of $\mathbb{F}_q$ are the roots of the equation $X^q - X$. The primitive $q-1$th roots of unity are precisely the elements which generate $\mathbb{F}_q$ and their minimal polynomial must be monic and irreducible of degree r. The number of primitive $q-1$th roots is $\phi(q-1)$ and so the number of points with residue field isomorphic to $\mathbb{F}_q$ is

$$\frac{\phi(q-1)}{r}.$$

2.13 (a) Suppose that X is Noetherian and let U be an open subset. Then U is Noetherian. But every Noetherian topological space is compact. Indeed if $\{U_i\}$ is an open cover of U, then consider all finite unions of elements U_I. If none of these is Lall of U then we can find an infinite increasing sequence of these, that is we can find an infinite decreasing sequence of closed sets.
Now suppose that every open subset is compact. Let

$$F_1 \supset F_2 \supset F_2, \dots,$$

be a decresing sequence of closed subsets. Let F be their intersection and let U be the complement of X. Then $U_i = U - F_i$ is an increasing sequence of open subsets of U, whose union is the whole of U. As U is compact, there is an index i such that $U = U_i$ and so the sequence stabilises at F_i.
(b) Suppose that $X = \operatorname{Spec} A$. Let $\{U_i\}$ be an open cover of X. Since open sets of the form U_f are a base for the topology, we may assume that $U_i = U_{f_i}$, for some $f_i \in A$. It is proved in the lectures that then a finite set of the U_i cover X.
(c) Suppose that $X = \operatorname{Spec} A$. A decreasing sequence of closed subsets corresponds to an increasing sequence of ideals in A. By assumption the set of ideals satisfies ACC so that the set of closed subsets satisfies DCC.
(d) The difference between R being Noetherian and $\operatorname{Spec} R$ being Noetherian is that the first says that the set of ideals satisfies ACC and the second says that the set of radical ideals satisfies ACC. So we want a ring with lots of ideals but not many radical ideals.
Let I be the ideal generated by $x_1 - x_2^2$, $x_2 - x_3^2$, $\dots$, inside the polynomial ring $k[x_1, x_2, \dots]$ localised at the ideal generated by $x_1, x_2, \dots$ and let R be the quotient ring. Then every element of R is equivalent to a polynomial in x_n, for some n and so every element of R is equivalent to a unit multiplied by a power of x_n. So the only non-trivial radical

ideal is the ideal containing $x_1, x_2, \ldots$. But the sequence of ideals

$$\langle x_1 \rangle \subset \langle x_2 \rangle \ldots,$$

is an increasing sequence of ideals.