1. Note that the coordinate ring of $A = A(X)$ is an integral domain if and only if $p$ is a prime ideal.
   So, suppose that $f$ and $g \in A$ and that $fg = 0$. Then
   
   $$ V(f) \cup V(g) = V(fg) = X. $$

   As $X$ is irreducible, either $V(f) = X$ or $V(g) = X$, that is either $f = 0$ or $g = 0$. Thus $A$ is an integral domain.

2. We first determine the image. We have $(x, y) = (t^2, t^3)$, so that $x = t^2$ and $y = t^3$. Clearly the image is contained in the zero set of $y^2 - x^3$. On the other hand, if $x \neq 0$, then set $t = y/x$, and note that $t^2 = y^2/x^2 = x$ and $t^3 = y^3/x^3 = y(y^2/x^3) = y$. If $x = 0$, then $y = 0$ and this is the image of 0. Thus the image is precisely $C = V(y^2 - x^3)$.

   Clearly this map defines a bijection $\pi$ between $\mathbb{A}^1$ and $C$. Since this map is a morphism, it is certainly continuous. Thus the only possible non-trivial closed subsets of the image are finite subsets. On the other hand, as any point is certainly closed, it follows that the closed subsets of $C$ are again either finite or the whole set. Thus $\pi$ is a homeomorphism.

   Now if $\pi$ were an isomorphism of varieties, then $A(\pi)$ would define an isomorphism of rings, between $A(C) = K[x, y]/\langle y^2 - x^3 \rangle$ and $K[t]$, carrying the maximal ideal $m = \langle x, y \rangle$ to the maximal ideal $n = \langle t \rangle$. This would induce an isomorphism of vector spaces between $n/n^2$ and $m/m^2$. But the former has dimension two (basis $x, y$) and the latter dimension one (basis $t$). So $A(\pi)$ is not an isomorphism.

3. (i) $U_x = \mathbb{A}^1 \setminus \{0\}$.

   (ii) Suppose that $X = V(f_1, f_2, \ldots, f_k)$. Give $\mathbb{A}^{n+1}$ coordinates $(x_1, x_2, \ldots, x_n, y)$ and let $Y = V(yf - 1, f_1, f_2, \ldots, f_k)$. Then the image of $Y$ under projection to the first factor is $U_f \subset X$ and this defines a morphism $\phi: Y \rightarrow U_f$. Define a morphism $U_f \rightarrow \mathbb{A}^{n+1}$ by the rule
   
   $$ (x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_n, 1/f). $$

   Then the image lies in $Y$ and the resulting morphism $\psi: X \rightarrow U_f$ is the inverse of $\phi$. Thus $Y \simeq U_f$ is affine. The coordinate ring of $Y$ is clearly equal to
   
   $$ A(X)[y]/\langle yf - 1 \rangle \simeq A(X)_f. $$
4. Suppose $Y$ is a closed subset of $\mathbb{A}^n$. If we are given a morphism $\phi: X \rightarrow Y$ then define a $K$-algebra homomorphism $\alpha(\phi): \mathcal{O}_Y(Y) \rightarrow \mathcal{O}_X(X)$ by the following rule:

Suppose we are given $f \in \mathcal{O}_Y(Y)$. Then we get a morphism $f: Y \rightarrow \mathbb{A}^1$ and composing we get a morphism $\alpha(\phi)(f) = f \circ \phi: X \rightarrow \mathbb{A}^1$.

It is clear that $\alpha(\phi)$ is a $K$-algebra homomorphism.

Now we define a map

$$\beta: \text{Hom}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)) \rightarrow \text{Hom}(X, Y),$$

as follows: Suppose that we are given a $K$-algebra homomorphism $a: \mathcal{O}_Y(Y) \rightarrow \mathcal{O}_X(X)$. Then let $\xi_1, \xi_2, \ldots, \xi_n$ be the restriction to $Y$ of the $n$ coordinate functions $x_1, x_2, \ldots, x_n$. Suppose that $f_i = a(\xi_i)$. Let $\beta(a): X \rightarrow Y$ be the morphism given by

$$x \mapsto (f_1(x), f_2(x), \ldots, f_n(x)).$$

We check that $\beta$ is the inverse of $\alpha$. Suppose we start with $\phi: X \rightarrow Y$. If we compose with $\xi_i$ then we get $f_i = \xi_i \circ \phi = \alpha(\phi)(\xi_i) \in \mathcal{O}_X(X)$. Therefore $\beta \circ \alpha(\phi)$ is the morphism given by

$$x \mapsto (f_1(x), f_2(x), \ldots, f_n(x)),$$

which is the same as $\phi$.

Now suppose we start with $a: \mathcal{O}_Y(Y) \rightarrow \mathcal{O}_X(X)$. We get a morphism $\phi$ given by

$$x \mapsto (f_1(x), f_2(x), \ldots, f_n(x)),$$

where $f_i = a(\xi_i)$. Then $\alpha(\phi)(\xi_i) = f_i$ as well. As $(\alpha \circ \beta)(a)$ and $a$ agree on the generators $\xi_1, \xi_2, \ldots, \xi_n$ they are the same $K$-algebra homomorphism.

Thus $\beta$ is the inverse of $\alpha$ and $\alpha$ is a bijection.

5. The coordinate rings of $\mathbb{A}^m$ and $\mathbb{A}^n$ are $K[x_1, x_2, \ldots, x_m]$ and $K[y_1, y_2, \ldots, y_n]$.

Note that the tensor product

$$K[x_1, x_2, \ldots, x_m] \otimes_K K[y_1, y_2, \ldots, y_n] \simeq K[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n],$$

is a finitely generated $K$-algebra without nilpotents. Since we have an equivalence of categories between affine varieties and finitely generated $K$-algebras without nilpotents, (4) implies the affine variety with coordinate ring

$$K[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n],$$

that is $\mathbb{A}^{m+n}$, is the product in the category of all quasi-projective varieties.
6. Clearly $K[x, y] = \mathcal{O}_{K^2}(\mathbb{A}^2) \subset \mathcal{O}_X(X)$. Now since $X \subset \mathbb{A}^2 - \{x = 0\}$, $\mathcal{O}_X(X) \subset K[x, y]_x$. Similarly $\mathcal{O}_X(X) \subset K[x, y]_y$. But then

$$\mathcal{O}_X(X) \subset K[x, y]_x \cap K[x, y]_y.$$ 

Suppose that $\alpha \in K[x, y]_x \cap K[x, y]_y$. Then there are $f$ and $g \in K[x, y]$ and $m$ and $n$ such that

$$\alpha = \frac{f}{x^m} = \frac{g}{y^n},$$

where this equality takes place in $K(x, y)$. By unique factorisation, we may assume that $f$ and $g$ are coprime to $x$ and $y$, respectively. Cross-multiplying, gives an equality in $K[x, y]$,

$$fy^n = gx^m,$$

so that $n = m = 0$. Thus

$$\mathcal{O}_X(X) \subset K[x, y]$$

and so we get equality. Now suppose that $X$ is affine. We have seen that the natural map

$$K[x, y] \longrightarrow A(X)$$

is an isomorphism. But then the inclusion $X \longrightarrow \mathbb{A}^2$ is an isomorphism, a contradiction.