MODEL ANSWERS TO HWK #5

1. The sheaf which assigns to every open set the zero group.
2. The morphism which assigns to every open set the zero map.
3. Note first that if $\phi\colon H \longrightarrow G$ is a group homomorphism then the kernel of ϕ is the kernel of ϕ in the category of groups. It is then clear that $\operatorname{Ker} f$ is the kernel in the category of presheaves. As $\operatorname{Ker} f$ is a sheaf if $\mathcal{F}$ and $\mathcal{G}$ are sheaves, it follows that $\operatorname{Ker} f$ is also the kernel in the category of sheaves.
4. (i) Note first that if $\phi\colon H \longrightarrow G$ is a group homomorphism then the cokernel of ϕ in the category of groups is equal to $G/\phi(H)$. The cokernel presheaf is then clearly the presheaf

$$U \longrightarrow \mathcal{G}(U)/f(\mathcal{F}(U)).$$

(ii) The cokernel sheaf is the sheaf $\operatorname{Coker} f$ associated to the cokernel presheaf. This satisfies the universal property of the cokernel sheaf, using the universal property of the cokernel presheaf and the universal property of the sheaf associated to the presheaf.
(iii) The exponential map between the sheaf of holomorphic functions and the sheaf of non-zero holomorphic functions given in 5 (v) below is surjective on stalks, so that the cokernel sheaf must be zero, since its stalks are all zero, but it is not surjective on global sections, so that the cokernel presheaf is not zero. There are many other such examples.
5. (i) Suppose that $\sigma \in \mathcal{F}$ is sent to zero. Then σ is in the image of zero, that is σ is zero.
(ii) Suppose that $\tau \in \mathcal{G}$. Then τ is sent to zero and so τ is in the image.
(iii) One direction is clear. If the map on sheaves is exact, then the map on stalks is exact. Now suppose that the map on stalks is exact. Then the map of sheaves $f_i \circ f_{i-1}$ is zero on stalks. But then it must be the zero map. It follows that $\operatorname{Ker} f_i \supset \operatorname{Im} f_{i-1}$. But since this inclusion is an equality stalk by stalk, it is an isomorphism, whence we have equality.
(iv) We may as well suppose that $U = X$. Let $\alpha\colon \mathcal{F} \longrightarrow \mathcal{G}$ and $\beta\colon \mathcal{G} \longrightarrow \mathcal{H}$. Consider the sheaf $\operatorname{Ker}\alpha$. As its stalks are all zero, $\operatorname{Ker}\alpha = 0$. But then $\operatorname{Ker}\alpha(X) = 0$, so that $\alpha(X)$ is injective. The composition $\beta \circ \alpha$ is zero, as it is zero on stalks. But then the image of $\alpha(X)$ is contained in the kernel of $\beta(X)$. Suppose that τ is in the kernel of $\beta(X)$. Then we may find an open cover U_i of X, such that $\tau_i = \tau|_{U_i}$ is the image of $\sigma_i \in \mathcal{G}(U_i)$. Since $\alpha(U_{ij})$ is injective and $\sigma_i|_{U_{ij}}$

and $\sigma_j|_{U_{ij}}$ are both sent to $\tau|_{U_{ij}}$ and $\mathcal{F}$ is a sheaf, it follows that there is a global section $\sigma \in \mathcal{F}(X)$ such that $\sigma|_{U_i} = \sigma_i$. Let τ' be the image of σ. Since $\tau'|_{U_i} = \tau_i$, it follows that $\tau' = \tau$.

(v) The classic example is the exponential sequence, say on the topological space $\mathbb{C}^* = \mathbb{C} - \{0\}$,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

Here $\mathcal{O}_X$ is the sheaf of holomorphic functions, $\mathcal{O}_X^*$ is the multiplicative group of non-zero holomorphic functions, $\mathbb{Z}$ is the sheaf of integer valued holomorphic functions (necessarily locally constant therefore) included into $\mathcal{O}_X$ and the second map takes the holomorhpic function f to the non-zero holomorphic function $e^{2\pi i f}$. The first map is surely injective and the kernel of the second map is surely $\mathbb{Z}$. The last map is surjective, since if $g \in \mathcal{O}_{X,p}^*$, then there is an open neighbourhood U of p for which $g \neq 0$. In this case the logarithm is well-defined and $f = \frac{1}{2\pi i} \log g$ is a holomorphic function mapping to g. Finally z is an element of $\mathcal{O}_X^*(X)$ which is not in the image.

There are many other examples.