MODEL ANSWERS TO HWK #4

1. (i) Suppose that $F_\alpha(X)$ are polynomials in the variables $X_1, X_2, \dots, X_n$, which define X. Then $F_\alpha(p) = 0$ for all α and so $p \in X$.
Let Λ' be the hyperplane $X_0 = 0$ and let $Y \subset \mathbb{P}^{n-1}$ be the closed subvariety defined by the homogeneous polynomials $F_\alpha(X)$. Suppose that $q \in Y$. Then $q = [0 : q_1 : q_2 : \dots : q_n]$ and any point r on the line joining p to q, not equal to p has the form $r = [q_0 : q_1, q_2, \dots, q_n]$. It is clear that $F_\alpha(r) = F_\alpha(q) = 0$.
(ii) First that if $\Lambda = \langle p, \Lambda_0 \rangle$ then taking the cone over Λ is the same as taking the cone over Λ_0 and then over p. So we might we as well assume that $\Lambda = \{p\}$ and use the same coordinates as in (i).
Then $Y \subset \Lambda'$ is defined by polynomials $F_\alpha(X)$ in the variables $X_1, X_2, \dots, X_n$ and these polynomials define X by (i).
2. The non-trivial closed sets are all finite.
3. The only closed subsets of $\mathbb{A}^1 \times \mathbb{A}^1$ are the whole space, the empty set, and finite unions of fibres of both projection maps and finitely many points. Indeed, this does define a topology, such that the projection maps are continuous, and clearly it is the smallest topology with this property.
On the other hand, in the category of varieties,

$$\mathbb{A}^1 \times \mathbb{A}^1 \simeq \mathbb{A}^2,$$

and the vanishing locus of the polynomial $xy - 1$ is not closed in the product topology.
4. A general bi-homogenous polynomial $F(X_0, X_1, Y_0, Y_1)$ of bi-degree $(1, 1)$ is of the form

$$F = aX_0Y_0 + bX_0Y_1 + cX_1Y_0 + dX_1Y_1.$$

Consider the associated matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The rank r of this matrix is either one or two. If it is one, then we may assume that (c, d) is a multiple of (a, b). In this case

$$F = (X_0 + \lambda X_1)(aY_0 + bY_1)$$

The zero locus of F is then the union of a fibre of both projections, and it is clear that we can change coordinates to get the required form.

Otherwise the rank is 2 and if we change coordinates via

$$(Y_0, Y_1) \longrightarrow (Y_0, aY_0 + bY_1),$$

then F reduces to

$$F = X_0Y_1 + X_1(cY_0 + dY_1).$$

As the rank is two, we may then change coordinates so that we get

$$F = X_0Y_1 - X_1Y_0,$$

which is clearly the equation of the diagonal.

5. If F is a bihomogenous polynomial of bi-degree $(1,2)$ then it has the form

$$X_0F_0 + X_1F_1,$$

where F_i are homogeneous quadratic polynomials in Y_0 and Y_1. Consider the vector space

$$W = \langle F_0, F_1 \rangle,$$

inside the space V of all homogeneous quadratic polynomials in Y_0 and Y_1. Note that a general change of coordinates of X_0 and X_1 corresponds to a change of basis of W.

Suppose that F_0 and F_1 are independent. Now $W \subset V$ defines a line $l \subset \mathbb{P}^2$. Inside $\mathbb{P}^2$, we have the locus of rank one quadratic forms (ie the pure powers). In coordinates

$$aY_0^2 + bY_0Y_1 + cY_1^2,$$

this locus is given by the discriminant

$$b^2 - 4ac,$$

which is a conic. The line l can either meet this locus in two points or one point. If it meets this locus in two points, then we may always change coordinates so that these points are Y_0^2 and Y_1^2, and we take this as our basis of W. This is the case of the twisted cubic.

Otherwise, the line meets the conic in one point. We may always choose this point to be Y_0^2, in which case the line is $c = 0$. But then we may choose a basis Y_0^2, Y_0Y_1. In this case we

$$F = Y_0(X_0Y_0 + X_1Y_1),$$

the equation of a conic union a line.

If F_0 and F_1 are not independent, then we may assume that $F_0 = F_1$. In this case

$$F = F_0(X_0 + X_1),$$

which after a change of coordinates has the form

$$X_0Y_0Y_1 \qquad \text{or} \qquad X_0Y_0^2.$$

In the first case we get a line union two skew lines. In the second a line union a double line.