1. If $G$ is a group, then let $C$ be the category with one object $\ast$, such that $\text{Hom}(\ast, \ast) = G$, with composition of morphisms given by the group law in $G$. Then the identity in $G$ plays the role of the identity morphism, the associative law in $G$ gives associativity of composition, and the existence of inverses in $G$ makes every morphism an isomorphism, and conversely.

2. Suppose that $U$ maps to both $W$ and $Z$ over $Y$. Then $U$ maps to both $W$ and $Z$ over $X$, as $f$ is a monomorphism. But then there is a unique morphism to $W \times^X Z$ by the universal property of the fibre product. But then $W \times^X Z$ satisfies the universal property of the fibre product over $Y$ and we are done by uniqueness of the fibre product.

3. (i) Clear.

(ii) The equaliser of $f_1$ and $f_2$: $X \rightarrow Y$ is the set 
\[ \{ x \in X \mid f_1(x) = f_2(x) \} \]
together with its natural inclusion into $X$.

(iii) Suppose that $C$ admits equalisers. Let $f: X \rightarrow B$ and $g: Y \rightarrow B$ be two morphisms. Then there are two morphisms $p: X \times Y \rightarrow B$ (respectively $q$), the composition of projection down to $X$ (respectively $Y$) and then $f$ (respectively $g$) as appropriate. Let $E$ be the equaliser of $p$ and $q$. Then $E$ maps to $X \times Y$, whence it maps to $X$ and $Y$, via either projection, and these two morphisms become equal when composed with $f$ and $g$. Now suppose that $Z$ maps to both $X$ and $Y$ over $B$. Then it maps to $X \times Y$, and composing with projection down to $X$ or $Y$ and then $f$ or $g$ as appropriate. It follows that $Z$ maps to $E$, by the universal property of the equaliser. But then $E$ satisfies the universal property of the fibre product.

Now suppose that $C$ admits fibre products. If $f$ and $g: X \rightarrow Y$ are two morphisms, then we get a morphism $X \rightarrow Y \times Y$, by definition of the product. Note that there is also a morphism $\delta: Y \rightarrow Y \times Y$ induced by the identity on both factors. Let $E = X \times_{Y \times Y} X$. Then $E$ maps to $X$ and composing this map with either $f$ or $g$ is the same. Suppose that $Z$ maps to $X$, such that the composition with $f$ or $g$ is the same. Then $Z$ maps to $X$ and its maps to $Y$ over $Y \times Y$. So $Z$ maps to $E$, by the universal property of the fibre product. But then $E$ satisfies the universal property of the equaliser.
(iv) Let $\mathcal{C}$ be the category with one object and two morphisms. Then nothing equalises the two morphisms, but fibre products do exist. Indeed if the two morphisms are $f$ and $g: X \to X$, then given any half of square, one can always fill it in with $X$ at the top, selecting $f$ and $g$ as appropriate to go on the two edges.

4. (i) Sketched in the lecture notes.

(ii) Let $I$ be an object of $\mathbb{I}$. By definition of $\alpha$, we are given a morphism $\alpha(I): F(I) \to G(I)$. Since $\lim_{\mathbb{I}} G$ is a prelimit, there are morphisms $\lim_{\mathbb{I}} G(I) \to \lim_{\mathbb{I}} G$. Composing, it follows that there are morphisms $\lim_{\mathbb{I}} F(I) \to \lim_{\mathbb{I}} G$. One can check easily that the construction of these morphisms is functorial with respect to morphisms $I \to J$ in $\mathbb{I}$.

By the universal property of the limit $\lim_{\mathbb{I}} F$, there is then a morphism $\lim_{\mathbb{I}} F \to \lim_{\mathbb{I}} G$.

The rest is tedious checking.

(iii) Suppose we are given morphisms from $T$ to $W$ and $X$ over $Z$. By composition, we are then given morphisms from $T$ to $W$ and $Y$ over $Z$. By the universal property of the fibre product, there is then a unique morphism $T \to Y \times_Z W$. So now we have morphisms from $T$ to $X$ and $Y \times W$ over $Y$. By the universal property of the fibre product, there is then a unique morphism $T \to X \times (Y \times W)$. So $X \times (Y \times W)$ satisfies the universal property of the fibre product. Alternatively one can proceed as follows. Let $\mathbb{I}$ be the category defining the fibre product. Let $F$ be the functor associated to $W$, $Y$ and $Z$ and let $G$ be the functor associated to $W$, $X$ and $Z$. Then there is a natural transformation $\alpha: G \to F$. It associates to the three objects of $\mathbb{I}$ the obvious maps $W \to W$, $X \to Y$ and $Z \to Z$. The isomorphism we are looking for is then given by

$$\lim (\alpha): \lim_{\mathbb{I}} F \to \lim_{\mathbb{I}} G.$$ 

5. Suppose that we are given a natural transformation $u: h_Y \to h_{Y'}$. Then we get a function $u_Y: h_Y(Y) \to h_{Y'}(Y)$. Let $\phi = u_Y(i_Y) \in h_{Y'}(Y)$. Then $\phi: Y \to Y'$ is a morphism.

It suffices to show that $u = h(\phi)$. If $X$ is an object of $\mathcal{C}$ then we get a function $u_X: h_Y(X) \to h_{Y'}(X)$. Pick $f \in h_Y(X)$. Since $u$ is a natural
transformation we get a commutative square

\[
\begin{array}{c}
Y \\ h_Y \downarrow \\
X \\ h_Y(X) \xrightarrow{u_X} h_Y'(X)
\end{array}
\]

Consider starting at the top left hand corner, with the morphism \( i_Y \). Going to the right and then down, we get \( f \circ \phi \). Going down and then to the right, we get \( u_X(f) \). Thus \( u_X(f) = f \circ \phi = h(\phi) \). Since \( f \) and \( X \) were arbitrary, the result follows.