MODEL ANSWERS TO HWK #2

1. Note that it suffices to prove that there is exactly one orbit of $\mathrm{PGL}(n+1)$ on the set of $n+2$ points in linear general position, and that the stabiliser of any point is trivial. In other words, we can fix the set of points $q_1, q_2, \dots, q_{n+2}$ and show that we can bring any other set of points $p_1, p_2, \dots, p_{n+2}$ to these points.
Now $n+2$ points in $\mathbb{P}^n$ correspond to $n+2$ vectors in the vector space K^{n+1}. Now any set of $n+1$ points in linear general position corresponds to a set of $n+1$ vectors which are linearly independent. Thus the first $n+1$ vectors are a basis of K^{n+1}. It follows that there is always an element of $\phi \in \mathrm{PGL}(n+1, K)$ carrying the first $n+1$ points to the standard set of $n+1$ points, given by the standard basis.
In other words we choose

$$q_0 = [1 : 0 : \dots 0], \quad q_1 = [0 : 1 : \dots 0], \quad \dots \quad q_n = [0 : 0 : \dots 0 : 1].$$

With this choice, the condition that q_{n+1} is independent is equivalent to the condition that none of its coordinates is zero. It is natural then to choose $q_{n+1} = [1 : 1 : \dots : 1]$. By assumption $p_i = q_i$, $i \leq n$ and by the same token as before, every coordinate of p_{n+1} is non-zero. Let A be a diagonal matrix, with non-zero determinant. Note that A fixes $q_0, q_1, \dots, q_n$. On the other hand if we pick the (i,i) entry to be the reciprocal of the ith entry of p_{n+1}, then we bring the point p_{n+1} to the point q_{n+1}.
It remains to check that the stabiliser of the $n+2$ points $q_0, q_1, \dots, q_{n+1}$ is trivial. Now the matrix A fixes the point $p = [v]$ iff v is an eigenvalue of A. But the only diagonal matrix which also fixes the last point is clearly a scalar matrix, which is equivalent to the identity matrix.
2. Let A be the corresponding matrix. Then A is classified up to conjugacy by its Jordan normal form. There are then three possibilities,

(1) A has two independent eigenvectors, with the same eigenvalue.
(2) A has two independent eigenvectors, with distinct eigenvalues.
(3) A has only one eigenvector.

In the first case, A is a scalar matrix. In this case ϕ is the identity map. In the second case, A is a diagonal matrix. In this case, we may always assume that the second eigenvector is one. It follows that ϕ has the form $\phi(z) = az$, where a is neither zero or one. Finally, in the case where A has a repeated eigenvalue, we may assume that A has the form

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Rescaling, we get a matrix with ones on the diagonal. This is a matrix with one eigenvector with eigenvalue one. In this case, it must be conjugate to the matrix above, but now with $\lambda = 1$. In this case $\phi(z) = z + 1$.

3. This is not hard however you do it, but it is almost completely trivial using the determinental description of the twisted cubic.

4. (a) Consider the matrix

$$M = \begin{pmatrix} X & Y & Z \\ Y & Z & W \end{pmatrix}.$$

We know that the points of the twisted cubic are precisely the points where this matrix has rank one. The minors correspond to the given quadratic polynomials. Suppose that we pick two of the three 2×2 minors. Then one column will belong to both minors.

By symmetry we may suppose that it is the first column. Now if both X and Y are non-zero, then Y is a non-zero multiple of X and by vanishing of the two minors, Y and Z and Z and W stand in the same relation. But then the first row is a multiple of the second and M is a matrix of rank one.

If $X = 0$ then the only way the first 2×2 minor is zero, is if $Y = 0$ as well. Thus we may as well assume that $Y = 0$. If $X \neq 0$ then both W and Z are zero, a point of the twisted cubic. Thus we may assume that $X = Y = 0$. In this case, clearly both 2×2 minors that contain the first column are zero, and the locus $X = Y = 0$ is a line. This line meets the cubic in one point, a tangent line.

(b) Now consider the matrix

$$A = \begin{pmatrix} \lambda_0 & -\lambda_1 & \lambda_2 \\ Z_0 & Z_1 & Z_2 \\ Z_1 & Z_2 & Z_3 \end{pmatrix}.$$

Expanding the determinant of A about the top row, we get the given linear combination of quadratic polynomials. Similarly for the matrix B,

$$B = \begin{pmatrix} \mu_0 & -\mu_1 & \mu_2 \\ Z_0 & Z_1 & Z_2 \\ Z_1 & Z_2 & Z_3 \end{pmatrix}.$$

Now the fact that the first determinant is zero implies that the row space of the first matrix is at most two dimensional. Similarly for the second matrix. Now suppose that we have a point of projective space that is common to the locus where A and B have rank at most two, but not a point of C. In this case the rank of M is two, and so the row rank of A and B is exactly two. It follows that the row spaces of A and B are equal and so spanned by the first two rows of each. Indeed both are in the row space, and they are independent by assumption. It follows that the matrices

$$C = \begin{pmatrix} \lambda_0 & -\lambda_1 & \lambda_2 \\ \mu_0 & -\mu_1 & \mu_2 \\ Z_0 & Z_1 & Z_2 \end{pmatrix},$$

and

$$D = \begin{pmatrix} \lambda_0 & -\lambda_1 & \lambda_2 \\ \mu_0 & -\mu_1 & \mu_2 \\ Z_1 & Z_2 & Z_3 \end{pmatrix},$$

have rank two, so their determinants are zero. Expanding these determinants gives two linear polynomials, in Z_0, Z_1, Z_2 and Z_3. Their common zero locus determines a line, and this line is easily seen to be either a secant line or a tangent line.

5. There are two ways to prove this. The first is geometric, the second algebraic. First the geometric. We may as well assume that we have $n+1$ points $p_1, p_2, \dots, p_{n+1}$ and it suffices to prove that these points don't lie in a hyperplane. Suppose they did. Suppose that the hyperplane is given as

$$\sum a_i Z_i = 0.$$

Pulling back to the curve, we would get

$$\sum a_i S^{n-i} T^i,$$

a homogeneous polynomial of degree n. By assumption this polynomial would have $n+1$ zeroes. But in this case the polynomial would be identically zero, a contradiction.

Now we turn to the algebraic proof. Lifting our $n+1$ points in $\mathbb{P}^n$ to $n+1$ vectors in K^{n+1}, we are given $n+1$ vectors, each of the form

$$(1, \lambda, \lambda^2, \dots, \lambda^n),$$

(the case where $\lambda = \infty$ can easily be reduced to this case). Taking determinants, it suffices to prove that the following determinant is non-zero,

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{vmatrix}.$$

In fact I claim that

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{vmatrix} = \prod_{i<j} (\lambda_i - \lambda_j),$$

whence the result.

This identity is well-known. Here is a quick sketch of the derivation of this identity. Note that both sides are zero, if $\lambda_i = \lambda_j$ (indeed the matrix on the left has a repeated column in this case). Suppose that we set $\lambda_i = x$. Since a polynomial is determined by its roots, up to scalars, and a polynomial of degree n can have at most n roots, the claim follows by induction, modulo checking the leading coefficient, which we leave as an exercise for the reader.

6. Clearly the image of $\mathbb{P}^n$ lies in the zero locus of these polynomials. This just says that

$$X^I X^J = X^{I'} X^{J'},$$

whenever $I + J = I' + J'$.

Now suppose we are given a point of $\mathbb{P}^N$ where each of these quadratic polynomials is zero. At least one of the coordinates of this point is non-zero. Define the length of an $(n+1)$-tuple I as the number of non-zero entries. Pick I_0 of minimal length such that $Z_I \neq 0$ (for notational convenience, I will temporarily drop the subscript from I_0).

Suppose that the length I is greater than one. Then we may find I' and J' such that

$$2I = I' + J',$$

and where one of I' and J' has smaller length than I. As

$$Z_I^2 = Z_{I'} Z_{J'},$$

this is a contradiction.

Possibly reordering, we may therefore assume that $I_0 = (d, 0, \dots 0) = de_0$. Let $I_i = (d-1)e_0 + e_i$, where $e_0, e_1, \dots, e_n$ is the standard basis of K^{n+1}. Set $X_i = Z_{I_i}$. I claim that $[Z_I]$ is the image of the point $[X_0 : X_1 : \cdots : X_n] \in \mathbb{P}^n$. Rescaling, we may assume that $X_0 = 1 = Z_{I_0}$ and it suffices to prove that

$$Z_I = X^I.$$

We prove this by an induction on the first coordinate of I. By assumption the result is true if $i_0 > d - 2$. Suppose it holds for all $i_0 > k$, where $k < d - 1$ and let I be an index such that $i_0 = k$. Then

$$Z_I Z_{I_0} = Z_{I'} Z_{J'}.$$

Here both I' and J' may be chosen with first coordinate greater than the average of

$$\frac{k+d}{2} \geq k + 1.$$

Thus, by induction,

$$Z_I = X_0^2 X^{I'} X^{J'} = X^{I'} X^{J'} = X^I.$$