1. Clearly a basis for the space of all homogeneous polynomials of degree \( d \) in \( n + 1 \) variables is given by the monomials. Thus it suffices to count the number of monomials. This is equivalent to the following problem in combinatorics. We are given \( n + 1 \) (marked) baskets and \( d \) (unmarked) balls, and we distribute the balls into the baskets. How many ways are there of doing this?

A very elegant solution to this problem involves a trick, known as stars and bars. Consider all sequences of \( d \) stars and \( n \) bars, for example, 
\[
* * | * * * || * .
\]

Thinking of the bars as the dividing lines between the baskets, there is an obvious correspondence between strings of stars and bars and ways to put balls into baskets (for example the string above corresponds to putting two balls into the first basket, three into the second, none into the third and one in the last).

Now any string of \( d \) stars and \( n \) bars has length \( n + d \) and the string is specified by the position of the bars (or indeed the stars). Thus the answer is equal to the number of ways of picking \( n \) objects from \( n + d \),
\[
\binom{n + d}{n} .
\]

For example, when \( d = 2 \) and \( n = 2 \), we get 
\[
\binom{4}{2} = 6 ,
\]

as expected.

2. Pick coordinates so that the line \( L \) is given by the equation \( Z = 0 \). Let \( F \) be the polynomial which defines \( C \), so that \( F \) is homogeneous of degree \( d \). By assumption \( C \cap L \) contains at least \( d + 1 \) points, so that the equation
\[
F(X, Y, 0) = 0
\]

has more than \( d \) roots. As a polynomial of degree \( d \) can only have at most \( d \) roots, it follows that \( F(X, Y, 0) \) is the zero polynomial. In this case we have
\[
F(X, Y, Z) = ZG(X, Y, Z),
\]

for some homogeneous polynomial of degree \( d - 1 \). Therefore the locus \( F = 0 \) contains the locus \( Z = 0 \), that is \( C \) contains \( L \).
3. Suppose that $X$ and $Y$ are two closed subvarieties of $\mathbb{P}^n$. By assumption there are homogeneous polynomials $\{F_\alpha\}$ and $\{G_\beta\}$ whose zero loci are $X$ and $Y$ respectively. Let $Z$ be the zero locus of $\{F_\alpha G_\beta\}$. If $x \in X$ then $F_\alpha(x) = 0$ for all $\alpha$, so that

$$F_\alpha(x)G_\beta(x) = 0,$$

for all $\alpha$ and $\beta$ and $x \in Z$. Thus $X \subset Z$ and by symmetry $X \cup Y \subset Z$. Now suppose that $w \notin X \cup Y$. Then we may find indices $\alpha$ and $\beta$ such that $F_\alpha(w) \neq 0$ and $G_\beta(w) \neq 0$. In this case

$$F_\alpha(w)G_\beta(w) \neq 0,$$

so that $w \notin Z$. Thus $Z = X \cup Y$ is a closed subvariety of $\mathbb{P}^n$.

Now suppose that $X_i$ are closed subvarieties of $\mathbb{P}^n$. If $\{F_{\alpha_i}\}$ are homogeneous polynomials defining $X_i$ then $\{F_{\alpha_i}\}_i$ (that is take the union over all $i$ and all $\alpha_i$) defines

$$\bigcap_i X_i.$$

Finally the homogeneous polynomial 1 shows that the emptyset is a closed subvariety and the homogeneous polynomial 0 shows that $\mathbb{P}^n$ is a closed subvariety.

Let $U$ be an open subset of $\mathbb{P}^n$. By assumption there are homogeneous polynomials $F_\alpha$ such that

$$U = \{ p \in \mathbb{P}^n \mid F_\alpha(p) \neq 0 \text{ for some } \alpha \}.$$

But then

$$U = \bigcup_\alpha U_{F_\alpha}.$$

4. This is easy, given (3). It suffices to prove that any point $p$ is closed. Changing coordinates we may assume that the point has coordinates $[1 : 0 : 0 : \cdots : 0]$, in which case the linear polynomials $X_1, X_2, \ldots, X_n$ define $p$. 