## 4. Categories and Functors

We recall the definition of a category:

**Definition 4.1.** *A category $\mathcal{C}$ is the data of two collections. The first collection is called the* ***objects*** *of $\mathcal{C}$ and is denoted $\mathrm{Obj}(\mathcal{C})$. Given two objects $X$ and $Y$ of $\mathcal{C}$, we associate another collection $\mathrm{Hom}(X, Y)$, called the* ***morphisms*** *between $X$ and $Y$. Further we are given a law of* ***composition*** *for morphisms: given three objects $X$, $Y$ and $Z$, there is an assignment*

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \longrightarrow \mathrm{Hom}(X, Z).$$

*Given two morphisms, $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, Z)$, $g \circ f \in \mathrm{Hom}(X, Z)$ denotes the composition. Further this data satisfies the following axioms:*

(1) *Composition is associative,*

$$h \circ (g \circ f) = (h \circ g) \circ f,$$

*for all objects $X$, $Y$, $Z$, $W$ and all morphisms $f\colon X \longrightarrow Y$, $g\colon Y \longrightarrow Z$ and $h\colon Z \longrightarrow W$.*

(2) *For every object $X$, there is a special morphism $i = i_X \in \mathrm{Hom}(X, X)$ which acts as an identity under composition. That is for all $f \in \mathrm{Hom}(X, Y)$,*

$$f \circ i_X = f = i_Y \circ f.$$

*We say that a category $\mathcal{C}$ is called* ***locally small*** *if the collection of morphisms is a set. If in addition the collection of objects is a set, we say that* ***small****.*

There are an abundance of categories:

**Example 4.2.** *The category (Sets) of sets and functions; the category of (Groups) groups and group homomorphisms; the category (Vec) of vector spaces and linear maps; the category (Top) of topological spaces and continuous maps; the category (Rings) of rings and ring homomorphisms. All of these are locally small categories.*

*Let $X$ be a topological space. We can define a small category $\mathfrak{Top}\, X$ associated to $X$ as follows. The objects of $\mathfrak{Top}\, X$ are simply the open subsets of $X$. Given two open subsets $U$ and $V$,*

$$\mathrm{Hom}(U, V) = \begin{cases} i_{UV} & \text{if } U \subset V \\ \emptyset & \text{otherwise.} \end{cases}$$

*Here $i_{UV}$ is a formal symbol. Composition of morphisms is defined in the obvious way (in fact the definition is forced, there are no choices to be made).*

**Definition 4.3.** *We say that a category $\mathcal{D}$ is a subcategory of $\mathcal{C}$ if every object of $\mathcal{D}$ is an object of $\mathcal{C}$ and for every pair of objects $X$ and $Y$ of $\mathcal{D}$, $\mathrm{Hom}_{\mathcal{D}}(X,Y)$ is a subset of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ (that is every morphism in $\mathcal{D}$ is a morphism in $\mathcal{C}$). The identity and composition of morphisms should come out the same.*

*We say that $\mathcal{D}$ is a full subcategory of $\mathcal{C}$, if for every pair of objects $X$ and $Y$ of $\mathcal{D}$, $\mathrm{Hom}_{\mathcal{D}}(X,Y)$ is equal to $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.*

The category of finite sets is a full subcategory of the category (Sets) of sets. Similarly the category of finite dimensional linear spaces is a full subcategory of the category (Vec) of vector spaces. By comparison the category (Groups) of groups is a subcategory of the category (Sets) of sets (somewhat abusing notation), but it is not a full subcategory. In other words not every function is a group homomorphism.

It is easy construct new categories from old ones:

**Definition 4.4.** *Given a category $\mathcal{C}$, the **opposite category**, denoted $\mathcal{C}^{\mathrm{op}}$, is the category, whose objects are the same as $\mathcal{C}$, but whose morphisms go the other way, so that*

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X).$$

**Definition 4.5.** *The **inverse** of a morphism $f\colon X \longrightarrow Y$ is a morphism $g\colon Y \longrightarrow X$, such that $f \circ g$ and $g \circ f$ are both the identity map. If the inverse of $f$ exists, then we say that $f$ is an **isomorphism** and that $X$ and $Y$ are **isomorphic**.*

**Definition 4.6.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A **covariant functor** from $F$ from $\mathcal{C}$ to $\mathcal{D}$ assigns to every object $X$ of $\mathcal{C}$ an object $F(X)$ of $\mathcal{D}$ and to every morphism $f\colon X \longrightarrow Y$ in $\mathcal{C}$ a morphism $F(f)\colon F(X) \longrightarrow F(Y)$ in $\mathcal{C}$, compatible with composition and the identity.*

*That is*

$$F(g \circ f) = F(g) \circ F(f) \qquad \text{and} \qquad F(i_X) = i_{F(X)}.$$

*A **contravariant functor** $F$ is the same as covariant functor, except that arrows are reversed,*

$$F(f)\colon F(Y) \longrightarrow F(X),$$

*and*

$$F(g \circ f) = F(f) \circ F(g).$$

In other words a contravariant functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is the same as a covariant functor $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}$

It is easy to give examples of functors. Let

$$F\colon (\underline{\text{Rings}}) \longrightarrow (\underline{\text{Groups}}),$$

be the functor which assigns to every ring $R$, the underlying additive group, and to every ring homomorphism $f$, the corresponding group homomorphism (the same map of course).

It is easy to check that $F$ is indeed a functor; for obvious reasons it is called a forgetful functor and there are many such functors.

Note that we may compose functors in the obvious way and that there is an identity functor. Slightly more interestingly there is an obvious contravariant functor from a category to its opposite.

There are three non-trivial well-known functors. First there is a functor, denoted $H_*$, from the category ($\underline{\text{Top}}$) of topological spaces to the category of (graded) groups, which assigns to every topological space its singular homology. Similarly there is a contravariant functor from category ($\underline{\text{Top}}$) of topological spaces to the category of (graded) rings, which assigns to every topological space its singular cohomology.

The second and third are much more general.

**Definition 4.7.** *Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. We say that $F$ is* ***faithful*** *if for every $f$ and $g$, morphisms in $\mathcal{C}$, $F(f) = F(g)$ iff $f = g$. We say that $F$ is* ***full*** *if for every morphism $h\colon F(X) \longrightarrow F(Y)$ in $\mathcal{D}$, there is a morphism $f$ in $\mathcal{C}$ such that $F(f) = h$. We say that $F$ is* ***essentially surjective*** *if for every object $A$ in $\mathcal{D}$ there is an object $X$ in $\mathcal{C}$ such that $A$ is isomorphic to $F(X)$.*

*We say that $F$ is an* ***equivalence of categories*** *if $F$ is fully faithful and essentially surjective.*

For example, let $\mathcal{D}$ be the category of finite dimensional vector spaces over a field $K$. Let $\mathcal{C}$ be the category whose objects are the natural numbers, and such that the set of morphisms between two natural numbers $m$ and $n$, is equal to the set of $m \times n$ matrices, with the obvious rule of composition. Then $\mathcal{C}$ is naturally a full subcategory of $\mathcal{D}$ (assign to $n$ the "standard" vector space $K^n$) and the inclusion map, considered as a functor, is an equivalence of categories. Note however that there is no functor the other way.

More generally, given a category $\mathcal{D}$, one may form a quotient category $\mathcal{C}$. Informally the objects and morphisms of $\mathcal{C}$ are equivalence classes of objects of $\mathcal{D}$, under isomorphism.

We defer the third definition. One of the more interesting notions of category theory, is the theory of limits.

**Definition 4.8.** *Let $\mathbb{I}$ be a category and let $F\colon \mathbb{I} \longrightarrow \mathcal{C}$ be a functor. A* ***prelimit*** *for $F$ is an object $L$ of $\mathcal{C}$, together with morphisms $f_I\colon L \longrightarrow F(I)$, for every object $I$ of $\mathbb{I}$, which are compatible in the following sense: Given a morphism $f\colon I \longrightarrow J$ in $\mathbb{I}$, the following diagram*

*commutes*

$$\begin{tikzcd} L \arrow[r, "f_I"] \arrow[dr, "f_J"'] & F(I) \arrow[d, "F(f)"] \\ & F(J). \end{tikzcd}$$

*The* ***limit of*** $F$*, denoted* $L = \varprojlim_{\mathbb{I}} F$ *is a prelimit* $L$*, which is universal amongst all prelimits in the following sense: Given any prelimit* $L'$ *there is a unique morphism* $g\colon L' \longrightarrow L$*, such that for every object* $I$ *in* $\mathbb{I}$*, the following diagram commutes*

$$\begin{tikzcd} L' \arrow[r, "g"] \arrow[dr, "f'_I"'] & L \arrow[d, "f_I"] \\ & F(I). \end{tikzcd}$$

Informally, then, if we think of a prelimit as being to the left of every object $F(I)$, then the limit is the furthest prelimit to the right. Note that limits, if they exist at all, are unique, up to unique isomorphism, by the standard argument. Note also that there is a dual notion, the notion of colimits. In this case, $F$ is a contravariant functor and all the arrows go the other way (informally, then, a prelimit is to the right of every object $F(I)$ and a limit is any prelimit which is furthest to the left).

Let us look at some special cases. First suppose we take for $\mathbb{I}$ the category with one object and one morphism. In this case a functor picks out an object. It is clear that in this case the limit is the same object. Similarly for the colimit.

It is in fact more interesting to take for $\mathbb{I}$ the empty category, that is the category with no objects and no morphisms. Then every object is a prelimit and so a limit has the property that every object has a unique map to it. For obvious reasons this is called a **terminal object**. The category (Sets) of sets has as terminal object any set with one object; the category (Vec) of vector spaces any space of dimension zero. The colimit has the property that it has a unique map to every object and it is called an **initial object**. The empty set is an initial object of the category (Sets) of sets; the group with one element is an initial object in the category (Groups) of groups.

At the other extreme one can take the identity functor, so that $\mathbb{I} = \mathcal{C}$. A limit, if it exists at all, is an object to which all other objects map (in a compatible fashion). In the case that a category has an initial object, then the limit of the identity functor is the initial object. Dually,

a colimit, if it exists at all, is an object which maps to all other objects. In the case that a category has a terminal object, then the colimit of the identity functor is the terminal object.

Now take as category two objects, with two morphisms (that is the two identity maps). A functor picks out two objects, call them $X$ and $Y$. First consider the case of the limit. A prelimit is the data of an object $Z$, together with a pair or morphisms, $f\colon Z \longrightarrow X$ and $g\colon Z \longrightarrow Y$. This prelimit is a limit iff it is universal amongst all such prelimits. That is suppose we are given two morphisms $f'\colon Z' \longrightarrow X$ and $g'\colon Z' \longrightarrow Y$, then there is a unique induced morphism $h\colon Z' \longrightarrow Z$, such that the following diagram commutes

$$\begin{tikzcd}
 & & X \\
Z' \arrow[urr, "f'"] \arrow[drr, "g'"'] \arrow[r, dashed] & Z \arrow[ur, "f"'] \arrow[dr, "g"] & \\
 & & Y.
\end{tikzcd}$$

Dually, consider the case of a colimit, where all the arrows are reversed. A prelimit is the data of an object $Z$, together with a pair of morphisms, $f\colon X \longrightarrow Z$ and $g\colon Y \longrightarrow Z$. This prelimit is a limit iff it is universal amongst all such prelimits. That is suppose we are given two morphisms $f'\colon X \longrightarrow Z'$ and $g'\colon Y \longrightarrow Z'$, then there is a unique induced morphism $h\colon Z \longrightarrow Z'$, such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[dr, "f"] \arrow[drr, "f'"] & & \\
 & Z \arrow[r, dashed] & Z'. \\
Y \arrow[ur, "g"] \arrow[urr, "g'"'] & &
\end{tikzcd}$$

**Definition 4.9.** *Let $X$ and $Y$ be two objects of a category $\mathcal{C}$. The* ***product*** *is the limit and the* ***coproduct*** *is the colimit, of the functor above.*

The product of two sets is the ordinary cartesian product; the product of two topological spaces is the product of the spaces and so on. The coproduct of two sets is their disjoint union; similarly for topological spaces; the coproduct of two vector spaces is the direct sum; similarly for groups and rings. Note that for groups, rings and vector spaces,

the coincidence that the product and coproduct are in fact isomorphic, even though they satisfy two quite different universal properties.

Now let us be a little more ambitious. Take a category with three objects and five morphisms. The two non-trivial morphisms should have the same domain, but different targets.

**Definition 4.10.** *Suppose we are given a diagram*

$$\begin{tikzcd} & Y \arrow[d, "g"'] \\ X \arrow[r, "f"] & B. \end{tikzcd}$$

*The limit of the corresponding functor, denoted* $X \underset{B}{\times} Y$*, is known as the* ***fibre product*** *or* ***fibre square****.*

As with the definition of the product, there is an accompanying commutative diagram

$$\begin{tikzcd} Z' \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\ & Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "g"'] \\ & X \arrow[r, "f"] & B. \end{tikzcd}$$

Note that if $B$ is a terminal object, then the fibre product is nothing more than the product.

**Lemma 4.11.** *The category (<u>Sets</u>) of sets admits fibre products.*

*Proof.* It is easy to check that

$$X \underset{B}{\times} Y = \{\, (x, y) \in X \times Y \mid f(x) = g(y) \,\},$$

does the trick. □

The fibre product is sometimes also known as the pullback. In other words we think of the morphism

$$X \underset{B}{\times} Y \longrightarrow X,$$

as the pullback of the map $g\colon Y \longrightarrow B$ along the map $f\colon X \longrightarrow B$. In particular the fibre of the former map over the point $x \in X$ is equal to the fibre of the map $g$ over the point $f(x)$.

The dual notion is that of pushout. Basically take the diagram above, flip about the $Y - X$-diagonal and reverse the arrows. Thus if we start with the diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & Y, \\ X & \end{tikzcd}$$

the pushout $Z$ has enjoys the universal property encoded in the following commutative diagram:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d] \arrow[ddr] & \\ X \arrow[r] \arrow[drr] & Z \arrow[dr, dashed] & \\ & & Z'. \end{tikzcd}$$

For example, consider the category of rings. Suppose we are given two ring homomorphisms $A \longrightarrow B$ and $A \longrightarrow C$, and two ring homomorphisms $B \longrightarrow P$ and $C \longrightarrow P$. Then we get a bilinear map $B \times C \longrightarrow P$, using multiplication in $P$. It is then easy to see that the pushout is the tensor product $B \underset{A}{\otimes} C$.

We now turn to the third important functor. We first note that given two categories $\mathcal{C}$ and $\mathcal{D}$, where $\mathcal{C}$ is locally small, the collection of all functors from $\mathcal{C}$ to $\mathcal{D}$ is a category, denoted $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. The objects of this category are simply functors from $\mathcal{C}$ to $\mathcal{D}$. Given two functors $F$ and $G$, a morphism between them is a natural tranformation:

**Definition 4.12.** *Let $F$ and $G$ be two functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. A* ***natural transformation*** *$u$ from $F$ to $G$ assigns to every object $X$ of $\mathcal{C}$ a morphism $u_X \colon F(X) \longrightarrow G(X)$ such that for every morphism $f \colon X \longrightarrow Y$ in $\mathcal{C}$ the following diagram commutes*

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "u_X"'] & F(Y) \arrow[d, "u_Y"'] \\ G(X) \arrow[r, "G(f)"] & G(Y). \end{tikzcd}$$

It is easy to check that we may compose natural transformations, that this composition is associative and that the natural transformation which assigns to every object $X$, the identity map from $F(X)$ to $F(X)$ acts as an identity, so that $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is indeed a category.

Suppose that $\mathcal{C}$ is a small category. Let $Y$ be an object of $\mathcal{C}$. I claim that we get a functor $h_Y \colon \mathcal{C} \longrightarrow (\underline{\text{Sets}})$. Given an object $X$ of $\mathcal{C}$, we associate the set $h_Y(X) = \text{Hom}(X, Y)$. Given a morphism $f \colon X \longrightarrow X'$, note that we get a map

$$h_Y(f) : \text{Hom}(X', Y) \longrightarrow \text{Hom}(X, Y),$$

which takes a morphism $g$ and assigns the morphism $h_Y(f)(g) = g \circ f$. It is easy to check that $h_Y$ is a contravariant functor. On the other hand, varying $Y$, I claim we get a functor

$$h \colon \mathcal{C} \longrightarrow \text{Fun}(\mathcal{C}^{\text{op}}, (\underline{\text{Sets}})).$$

At the level of objects, the definition of this functor is obvious. Given $Y \in \mathcal{C}$ we assign the object $h_Y \in \text{Fun}(\mathcal{C}^{\text{op}}, (\underline{\text{Sets}}))$. On the other hand, given a morphism $f \colon Y \longrightarrow Y'$, I claim that we get a natural transformation $h(f)$ between the two functors $h_Y$ and $h_{Y'}$ going from $\mathcal{C}^{\text{op}}$ to $(\underline{\text{Sets}})$. Thus given an object $X$ in $\mathcal{C}$, we are supposed to give a morphism

$$h(f)_X \colon h_Y(X) = \text{Hom}(X, Y) \longrightarrow h_{Y'}(X) = \text{Hom}(X, Y').$$

The definition of $h(f)_X$ is clear. Given $g \in \text{Hom}(X, Y)$, send this to $h(f)_X(g) = f \circ g$. It is easy to check that $h(f)$ is indeed a natural transformation and that $h$ is a functor. More significantly:

**Theorem 4.13** (Yoneda's Lemma)**.** *$h$ is fully faithful.*

The proof is left as an exercise for the reader. Yoneda's Lemma thus says that if we want to understand the category $\mathcal{C}$, we can think of it as a subcategory of the category of contravariant functors from $\mathcal{C}$ to the category $(\underline{\text{Sets}})$ of sets.

In these terms obviously the must fundamental question is to ask which of these functors is in the image:

**Definition 4.14.** *We say that the functor $F \colon \mathcal{C}^{\text{op}} \longrightarrow (\underline{\text{Sets}})$ is* ***representable*** *(by $Y$) if it is isomorphic to $h_Y$, for some object $Y$ of $\mathcal{C}$.*

By Yoneda's Lemma, if $F$ is representable by $Y$ then $Y$ is determined up to unique isomorphism.