3. Morphisms between varieties I

We adopt the following working definition of a morphism between varieties.

**Definition 3.1. A morphism**

\[ f : V \rightarrow W, \]

between two affine varieties, where \( V \subset \mathbb{A}^m \) and \( W \subset \mathbb{A}^n \) is given by picking a collection of \( n \) polynomials \( f_1, f_2, \ldots, f_n \in K[x_1, x_2, \ldots, x_m] \) such that

\[ f(x) = (f_1(x), f_2(x), \ldots, f_n(x)) \in W, \]

for every \( x \in V \).

A morphism

\[ f : V \rightarrow W, \]

between two projective varieties, where \( V \subset \mathbb{P}^m \) and \( W \subset \mathbb{P}^n \) is given by picking a collection of \( n \) homogenous polynomials \( F_0, F_1, \ldots, F_n \in K[X_0, X_1, \ldots, X_m] \) such that

\[ f(x) = [F_0(x) : F_1(x) : \cdots : F_n(x)] \in W, \]

for every \( x \in V \), where for every \( x \in V \) there is an \( i \) such that \( F_i(x) \neq 0 \).

Note that this gives us two categories. The category of affine varieties, with maps given by affine morphisms and the category of projective varieties, with maps given by projective morphisms.

Observe that if \( w = \lambda v \), and each \( F_i \) has degree \( d \), then

\[ F_i(w) = \lambda^d F_i(v), \]

so that

\[ [F_0(w) : F_1(w) : \cdots : F_n(w)] = [\lambda^d F_0(v) : \lambda^d F_1(v) : \cdots : \lambda^d F_n(v)] = [F_0(v) : F_1(v) : \cdots : F_n(v)]. \]

**Example 3.2.** The map

\[ f : \mathbb{P}^1 \rightarrow \mathbb{P}^2, \]

given by

\[ [S : T] \rightarrow [S^2 : ST : T^2], \]

is a morphism. Indeed we only need to check that \( S^2, ST \) and \( T^2 \) cannot be simultaneously zero, which is clear.
It is interesting and instructive to look at the image. Suppose that we pick coordinates $[X : Y : Z]$ on $\mathbb{P}^2$. On the image we have

\[
Y^2 = (ST)^2 \\
= S^2T^2 \\
= XZ.
\]

Thus the image lies in the locus $Y^2 - XZ = 0$. On the other hand suppose we have a point $[X : Y : Z] \in \mathbb{P}^2$. If $X \neq 0$, set $S = X$ and $T = Y$. Then

\[
[S^2 : ST : T^2] = [X^2 : XY : Y^2] \\
= [X^2 : XY : XZ] \\
= [X : Y : Z],
\]
as $X \neq 0$. We need to worry about the case $X = 0$. One way to proceed is to observe that then $Y = 0$ so that we have the point $[0 : 0 : 1]$, the image of $[0 : 1]$. Or observe that $X$ and $Z$ cannot simultaneously be zero, and if $Z \neq 0$, we set $S = Y$ and $T = Z$ and argue as before (using the obvious symmetry). Either way we have established that the image of $\mathbb{P}^1$ is a conic in $\mathbb{P}^2$.

It is interesting to see what happens when we dehomogenise. Suppose that $S \neq 0$. Then $X = S^2 \neq 0$. Then we get

$$\mathbb{A}^1 \longrightarrow \mathbb{A}^2$$

given as,

$$t \longrightarrow (t, t^2),$$

where $t = T/S$, and $y = Y/X = t$, $z = Z/X$. It is easy to see that the image has equation $y^2 = z$, which is course the dehomogenisation of $Y^2 = XZ$.

This example has many interesting generalisations. For example we can look at the morphism

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^3,$$

given as

$$[S : T] \longrightarrow [S^3 : S^2T : ST^2 : T^3].$$

If we dehomogenise, we get

$$t \longrightarrow (t, t^2, t^3),$$

which is often seen in calculus courses. The image $C$ is known as the **twisted cubic**.

It is interesting and instructive to try and write down equations for the twisted cubic. We have $[X : Y : Z : W] = [S^3 : S^2T : ST^2 : T^3].$
Thus certainly $Y^2 = XZ$, $XW = YZ$ and $Z^2 = YW$. It is an interesting exercise to check that these equations define the image.

More generally still, we can look at the morphism

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^d,$$

given as

$$[S : T] \longrightarrow [S^d : S^{d-1}T : \cdots : ST^{d-1} : T^d].$$

The image is called a **rational normal curve of degree** $d$.

More generally still, there is a morphism

$$\mathbb{P}^n \longrightarrow \mathbb{P}^N,$$

given as

$$[X_0 : X_1 : \cdots : X_n] \longrightarrow [X^I],$$

where given an $n$-tuple $I = (i_0, i_1, \ldots, i_n)$, $X^I$ denotes the monomial $X_0^{i_0}X_1^{i_1}\cdots X_n^{i_n}$. Here we choose coordinates $Z_I$, where $I$ ranges over all $n$-tuples of positive integers, whose sum is $d$. $N$ is equal to the number of such $n$-tuples, minus one. Note that not every $X^I$ can be zero. Indeed if $X_0^d = X_1^d = \cdots X_n^d = 0$, then $X_0 = X_1 = \cdots = X_n = 0$. This morphism is called the $d$-**uple embedding**.

Note that for every $I$, $J$, $I'$ and $J'$ such that $I + J = I' + J'$, the image lies in the hypersurface

$$Z_I Z_J = Z_{I'} Z_{J'},$$

since

$$X^I X^J = X^{I+J} = X^{I'+J'} = X^{I'} X^{J'}.$$

Once again, in fact the image is cut out by these equations.

Perhaps the most interesting example is to take $n = d = 2$. In this case we get a morphism

$$\mathbb{P}^2 \longrightarrow \mathbb{P}^5,$$

given as


$$[Z_{(2,0,0)} : Z_{(0,2,0)} : Z_{(0,0,2)} : Z_{(0,1,1)} : Z_{(1,0,1)} : Z_{(1,1,0)}].$$

This morphism is called the **Veronese morphism** and the image is called the **Veronese surface**. It turns out that the Veronese surface is an exception to practically every (otherwise) general statement about projective varieties.
Definition 3.3. PGL\(_n(K)\) denotes the space of invertible \(n \times n\) matrices with entries in \(K\), modulo the normal subgroup of scalar matrices, that is

\[
PGL_n(K) = \frac{GL_n(K)}{K^*}.
\]

Note that the canonical action of GL\(_{n+1}(K)\) on \(K^{n+1}\) descends to an action of GL\(_{n+1}(K)\) on \(\mathbb{P}^n\), in an obvious way. Clearly the set of scalar matrices acts trivially and in fact it is not hard to see that the scalar matrices are the kernel of the induced homomorphism. On the other hand, it is also easy to see that if we fix a matrix \(A\), then the induced bijection

\[
\mathbb{P}^n \longrightarrow \mathbb{P}^n
\]

is in fact a morphism. Thus the group PGL\(_n(K)\) is a subgroup of the group of all automorphisms of \(\mathbb{P}^n\).

It is interesting to see what happens for \(\mathbb{P}^1\). Suppose we take a matrix

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Then \(A\) sends \([X : Y]\) to

\[
[aX + bY : cX + dY].
\]

Suppose we work in the affine chart \(z = X/Y\). Then \(A\) sends \(z\) to

\[
\frac{aX + bY}{cX + dY} = \frac{a(X/Y) + b}{c(X/Y) + d} = \frac{az + b}{cz + d}.
\]

In the case, when \(K = \mathbb{C}\), we recover the Möbius group, the group of Möbius transformations.

Perhaps the most interesting property of PGL\(_n(K)\) is the following:

Theorem 3.4. Let \(p_1, p_2, \ldots, p_{n+2}\) and \(q_1, q_2, \ldots, q_{n+2}\) be two sets of \(n + 2\) in \(\mathbb{P}^n\) in linear general position.

Then there is a unique element of \(\phi \in PGL_n(K)\) such that

\[
\phi(p_i) = q_i.
\]

Using this, in the case \(n = 1\), we can give a synthetic construction of the unique conic through five points \(p_1, p_2, p_3, p_4\) and \(p_5\) in linear general position, which is known as the Steiner construction. Fix two points \(p = p_1\) and \(q = p_2\). Consider the set of lines through \(p\).

Definition 3.5. Suppose that \(\mathbb{P}^n = \mathbb{P}(V)\). Then \(\hat{\mathbb{P}}^n = \mathbb{P}(V^*)\) is called the dual projective space.
The whole point of \( \mathbb{P}^n \) is that it parametrises hyperplanes in \( \mathbb{P}^n \). Indeed an element of \( V^\ast \) is a linear functional on \( V \). Its zero locus is a hyperplane in \( V \) and this defines a hyperplane in \( \mathbb{P}^n \). Conversely a hyperplane in \( \mathbb{P}^n \) corresponds to a hyperplane in \( V \). This defines a linear functional on \( V \), up to scalars, that is an element of \( \mathbb{P}(V^\ast) \).

Another way of putting this is as follows. Pick coordinates \( X_0, X_1, \ldots, X_n \) on \( V \). These form a basis of \( V^\ast \). A general element of \( V^\ast \) is then of the form

\[
a_0X_0 + a_1X_1 + \cdots + a_nX_n,
\]

and its zero locus is a hyperplane in \( \mathbb{P}^n \).

**Lemma 3.6.** Let \( \Lambda \subset \mathbb{P}^n \) be a linear subspace of \( \mathbb{P}^n \) of dimension \( k \).

Then the set, of linear spaces \( \Gamma \) of dimension \( k+1 \) (or of dimension \( n-1 \)) containing \( \Lambda \), is a copy of projective space of dimension \( n-k-1 \).

**Proof.** We will give three different proofs of this result and we will also show that these two cases are duals of each other.

The first is geometric. Pick \( \Lambda' \) a complimentary linear subspace (that is \( \Lambda' \) has the property that it is disjoint from \( \Lambda \) and of maximal dimension with this property). Then \( \Lambda' \) is of dimension \( n-k-1 \), so that it is a copy of projective space of dimension \( n-k-1 \).

**Claim 3.7.** The points of \( \Lambda' \) are in bijection with linear spaces of dimension \( k+1 \) containing \( \Lambda \).

**Proof of (3.7).** One direction is clear. Given a point of \( \Lambda' \) the span of this point and \( \Lambda \) is a linear space of dimension \( k+1 \) containing \( \Lambda \).

On the other hand, a linear space \( \Gamma \) of dimension \( k+1 \) containing \( \Lambda \) must meet \( \Lambda' \) in a unique point. Indeed the dimension of the intersection of \( \Gamma \) and \( \Lambda' \) is at least zero. On the other hand, if it were positive dimensional, then there would be a line \( l \) in the intersection. This line is contained in \( \Gamma \) and \( \Lambda \) is a hyperplane in \( \Gamma \), so that \( l \) and \( \Lambda \) must meet in a point. But this contradicts the fact that \( \Lambda \) and \( \Lambda' \) are disjoint. \( \Box \)

The second is algebraic. Pick coordinates so that \( \Lambda \) is given as \( Z_{k+1} = \ldots = Z_n = 0 \). Then a hyperplane containing \( \Lambda \) is given by an equation of the form

\[
a_{k+1}Z_{k+1} + \cdots + a_nZ_n = 0.
\]

Thus the set of hyperplanes containing \( \Lambda \) is naturally in bijection with \( \mathbb{P}^{n-k-1} \), with coordinates \([a_{k+1} : a_{k+2} : \cdots : a_n] \).

The third uses a little linear algebra. Suppose that \( \mathbb{P}^n = \mathbb{P}(V) \).

Then \( V \) has dimension \( n+1 \) and \( \Lambda = \mathbb{P}(W) \), where \( W \) is of dimension \( k+1 \). Suppose that \( \Gamma = \mathbb{P}(U) \). Then the set of \( U \) containing \( W \) is in bijection with the set of \( U' \) of dimension one in \( V/W \). But the
latter is by definition \( \mathbb{P}(V/W) \) and as \( V/W \) has dimension \( n - k \), the result follows. By duality, hyperplanes in \( \mathbb{P}(V/W) \) correspond to lines in \( \mathbb{P}(V^*/W^*) \) and so the two results are indeed dual. \( \square \)

Thus the set of lines \( H_p \) through \( p \) is naturally a copy of \( \mathbb{P}^1 \). Similarly for the set \( H_q \) of lines through \( q \). Choose parametrisations \( L_t \) and \( M_t \) of these set of lines. Formally we pick isomorphisms \( \mathbb{P}^1 \longrightarrow H_p \) and \( \mathbb{P}^1 \longrightarrow H_q \). The two lines \( L_t \) and \( M_t \) intersect in a point \( p_t \). Varying \( t \), the locus of points \( p_t \) sweeps out a curve, call it \( C \). First note that \( C \) contains \( p \) and \( q \), provided that the line \( \langle p, q \rangle \) does not correspond to the same parameter value (we will check later that our choice of parametrisations satisfies this condition).

Note that we have three degrees of freedom left. Indeed we may choose our parametrisation of \( H_p \) so that \( t = 0 \) corresponds to the line \( \langle p, p_3 \rangle \), \( t = 1 \) to \( \langle p, p_4 \rangle \) and \( t = \infty \) to \( \langle p, p_5 \rangle \), using (3.4). Similarly for \( H_q \). It follows then that \( C_t \) passes through \( p_3, p_4 \) and \( p_5 \).

It remains to check that \( C \) is a conic. There are two ways to see this. The first is by direct computation. If \( L_t \) is given by \( aX + bY + cZ \) and \( M_t \) is given by \( dX + eY + fZ \) then the point of intersection of \( L_t \) and \( M_t \) has equation

\[
\begin{vmatrix}
a & b & c \\
d & e & f \\
x & y & z 
\end{vmatrix} = 0.
\]

Expanding about the last row, this clearly gives a quadratic equation in the variables \( a-f \). These are in turn linear in \( S \) and \( T \), so we get three quadratic polynomials \( F, G \) and \( H \). \( C \) is then the image of the morphism

\[
\mathbb{P}^1 \longrightarrow \mathbb{P}^2,
\]

given by

\[
[S : T] \longrightarrow [F : G : H].
\]

It is now easy to see that \( C \) is a conic:

**Lemma 3.8.** Let \( C \subset \mathbb{P}^d \) be the image of a morphism

\[
[S : T] \longrightarrow [F_0 : F_1 : \cdots : F_d],
\]

where \( F_0, F_1, \ldots, F_d \) have degree at most \( d \).

If \( C \) is not contained in a hyperplane then \( C \) is projectively equivalent to a rational normal curve of degree \( d \).

**Proof.** Note that \( C \) is not contained in a hyperplane if and only if \( F_0, F_1, \ldots, F_d \) are linearly independent. Since the space of homogeneous polynomials of degree \( d \) in \( S \) and \( T \) has dimension \( d + 1 \), it follows that the polynomials \( F_0, F_1, \ldots, F_d \) are a basis for the homogeneous
polynomials of degree $d$. But then we may find a linear transformation taking $S^iT^{d-i}$ to $F_i$, that is an element of $\text{GL}(d)$, and this defines a projective equivalence with the rational normal curve of degree $d$. □

Note that in our case if $C$ were to lie in a hyperplane then it would be a line, which is not the case, since $C$ contains $p_1, p_2, p_3, p_4$ and $p_5$ and any three of these points are not collinear.

Actually there is another way to check that $F$, $G$ and $H$ have degree 2. The basic idea is that to find the degree of a curve $C$, just intersect with a typical line $L$. The number of points $|C \cap L|$ will just be the degree of the curve. In fact if the line is given by $aX + bY + cZ$ then we just need to find the solutions to the equation

$$aF + bG + cH = 0.$$ 

If $F$, $G$ and $H$ have degree $d$ then this equation ought to have $d$ solutions. Borrowing a result from later in the course, in fact we can always choose $L$ with the property that $|L \cap C| = d$ (this is equivalent to saying that not every line is a tangent line).

So pick a typical line $L$ (in particular a line that does not contain any of $p_1, p_2, p_3, p_4$ and $p_5$). We calculate $|C \cap L|$.

Since the set $L$ is in (non-canonical) bijection with $H_p$ and $H_q$, a moments thought will convince the reader that picking $L$ determines an automorphism $\phi \in \text{PGL}(2)$ (indeed send $L_t \cap L$ to $M_t \cap L$) and we want to calculate the number of fixed points of $\phi$.

**Lemma 3.9.** Let $\phi \in \text{PGL}(2)$.

Then $\phi$ is conjugate to

1. The identity,
2. $z \rightarrow az$, some $a \in K^*$,
3. $z \rightarrow z + 1$.

Moreover the three cases are determined by the number of fixed points; at least three; two, one.

Thus the degree of $F$, $G$ and $H$ is at most 2. Now suppose that the line $\langle p, q \rangle$ corresponds to the same parameter value. In this case $d \leq 1$ since one of the fixed points of $\phi$ corresponds to $L \cap \langle p, q \rangle$. But then $C$ would be a line, which contradicts the fact that it contains $p_3, p_4$ and $p_5$. Another way to proceed, which generalises better to higher dimensions, is to consider the line $H = \langle p_3, p_4 \rangle$. We get an automorphism of this line, by sending the point $L_t \cap H$ to $M_t \cap H$. This automorphism would have three fixed points, $p_3$, $p_4$ and $H \cap \langle p, q \rangle$. But then this automorphism would be the identity. This can only happen if $C = H$ and so $p_5$ would also lie on $H$, a contradiction.
This result has the following interesting generalisation:

**Theorem 3.10.** Let \( p_1, p_2, \ldots, p_{n+3} \) be \( n + 3 \) points in linear general position in \( \mathbb{P}^n \).

Then there is a unique rational normal curve through these points.

**Proof.** We will do the case of \( n = 3 \) (the general case is no harder, just notationally more involved). Let \( l \) be the line \( \langle p_1, p_2 \rangle \), \( m \) be the line \( \langle p_2, p_3 \rangle \) and \( n \) the line \( \langle p_1, p_3 \rangle \). The set of planes that contains \( l \), \( H_l \), is a copy of \( \mathbb{P}^1 \). Pick three parametrisations of the three copies of \( \mathbb{P}^1 \), \( H_l \), \( H_m \) and \( H_n \). We choose these parametrisations subject to the condition that the line plane spanned by \( p_1, p_2 \) and \( p_3 \) corresponds to three different parameter values. Given \( t \in \mathbb{P}^1 \), the three planes corresponding to \( t \) intersect in a point, and so we get a curve \( C \) in \( \mathbb{P}^3 \).

Once again we have three degrees of freedom. We may choose our parametrisations, so that \( t = 0 \) corresponds to the three planes \( \langle l, p_4 \rangle \), \( \langle m, p_4 \rangle \) and \( \langle n, p_4 \rangle \). In this way, we may pick \( C \) so that it contains the six points \( p_1, p_2, \ldots, p_6 \) (we check our non-degeneracy condition at the end).

It remains to check that \( C \) is a twisted cubic. As before we could use Cramer’s rule to conclude that \( C \) is the image of

\[
[S : T] \rightarrow [F : G : H : I],
\]

where \( F, G, H \) and \( I \) all have degree three and then we just apply (3.8).

Instead, we could also the geometric argument. As before, it suffices to check that \( C \) meets a general plane \( P \) in three points. We use the same argument. The planes \( P_t \) and \( Q_t \) intersect \( P \) in a single point \( x \). Similarly the planes \( Q_t \) and \( R_t \) intersect \( P \) in a single point \( y \). \( H \) intersects \( C \) at the point corresponding to \( t \) iff \( x = y \). The assignment \( x \rightarrow y \) is an automorphism of \( P \), a copy of \( \mathbb{P}^2 \), and any automorphism of \( \mathbb{P}^2 \), not equal to the identity, can have at most three fixed points.

Now suppose that \( P_{t_0} = Q_{t_0} \) (necessarily \( \langle p_1, p_2, p_3 \rangle \)), for some \( t_0 \). Consider the line \( L = \langle p_4, p_5, \rangle \). The automorphism given by sending \( P_t \cap L \) to \( Q_t \cap L \) would have three fixed points, \( p_3, p_4 \) and \( L \cap \langle p_1, p_2, p_3 \rangle \). But then \( p_6 \) must also lie on \( L \), a contradiction. \( \square \)

Finally it seems worthwhile to point out that there are other ways to construct rational normal curves.

**Definition 3.11.** Let \( k \) be a positive integer. A subset \( X \) of projective space is a **determinental variety** if \( X \) is the locus where a matrix \( M = (F_{ij}) \) of homogeneous polynomials \( F_{ij} \) has rank at most \( k \).
For example, consider the matrix
\[
\begin{pmatrix}
X_0 & X_1 & X_2 & \ldots & X_n - 1 \\
X_1 & X_2 & \ldots & X_{n-1} & X_n
\end{pmatrix}
\]

The locus where this matrix has rank one is precisely a rational normal curve. Indeed if
\[
[X_0 : X_1 : \cdots : X_n] = [S^n : S^{n-1}T : \cdots : T^n],
\]
then clearly the second row is nothing more than the first row times \(T/S\). Conversely if the given matrix has rank 1, then the second row is a scalar multiple of the first, and it is easy to get the result.

On the other hand, the locus where a matrix has rank at most one, is exactly the locus where the \(2 \times 2\) minors are all zero. For example, for \(n = 3\), we recover the three quadrics containing a twisted cubic.

We can do a similar thing for the Veronese. In this case, we look at the locus where the matrix
\[
\begin{pmatrix}
Z_{2,0,0} & Z_{1,1,0} & Z_{1,0,1} \\
Z_{1,1,0} & Z_{0,2,0} & Z_{0,1,1} \\
Z_{1,0,1} & Z_{0,1,1} & Z_{0,0,2}
\end{pmatrix},
\]
has rank one.