2. Conics in $\mathbb{P}^2$

One of the beautiful results of classical projective geometry is the following:

Lemma 2.1. *Let $f \in \mathbb{R}[x, y]$ be a polynomial of degree two. Suppose that $f = 0$ contains more than one real point. Let F be the homogenisation of f.*
   Then $f = 0$ is a circle iff $F = 0$ contains the points $[1 : \pm i : 0]$.

Proof. Suppose that $f = 0$ defines a circle. Then $f(x, y)$ has the form

$$(x - a)^2 + (y - b)^2 = r^2.$$

Thus F is equal to

$$(X - aZ)^2 + (Y - bZ)^2 = r^2 Z^2.$$

Set $Z = 0$. Then $X^2 + Y^2 = 0$, which has the solution $[1 : \pm i : 0]$.
   Conversely suppose that $F = 0$ contains the points $[1 : \pm i : 0]$. Then

$$F(X, Y, 0) = aX^2 + bXY + cY^2,$$

vanishes at $[1 : \pm i : 0]$. Thus

$$ax^2 + bx + c = 0,$$

has roots $\pm i$, which is only possible if $b = 0$ and $a = c$. Hence $F(X, Y, 0)$ is a non-zero multiple of $X^2 + Y^2$. Possibly rescaling, we may assume that

$$F(X, Y, Z) = X^2 + Y^2 + ZG(X, Y, Z)$$

where $G(X, Y, Z)$ is a linear polynomial. Thus

$$f(x, y) = x^2 + y^2 + g(x, y),$$

for some linear polynomial g. Completing the square, we can put this in the form

$$(x - a)^2 + (y - b)^2 = k.$$

The condition that $f = 0$ contains more than one point is equivalent to requiring that $k > 0$, so that $k = r^2$, some $r > 0$ and we have the equation of a circle. $\qquad\square$

Since we want to work over $\mathbb{C}$, it turns out that we want to reinvent the wheel:

Definition 2.2. *The curve $C \subset \mathbb{P}^2_{\mathbb{C}}$, given as $F = 0$, is a **circle** if F has degree two and C contains the points $[1 : \pm i : 0]$.*

Let us consider the general polynomial of degree two in X, Y and Z,
$$F(X, Y, Z) = aX^2 + bY^2 + cZ^2 + dYZ + eXZ + fXY,$$
where a, b, c, d, e and f are in K. Thus polynomials of degree two are naturally in correspondence with K^6. On the other, if $F = \lambda G$, $\lambda \neq 0$, then F and G define the same zero locus. Over an algebraically closed field, the converse is true. Thus the set of conics in $\mathbb{P}^2$ is naturally in bijection with K^6 modulo scalars, that is $\mathbb{P}^5$.

Given that we want to count how many circles pass through two points and that a circle is nothing more than a conic that passes through two fixed points, the natural problem is to identify the following locus:
$$H_p = \{\, [a : b : c : d : e : f] \in \mathbb{P}^5 \,|\, F = 0 \text{ passes through } p \,\},$$
where $p \in \mathbb{P}^2$ is a point.

Lemma 2.3. *$H_p \subset \mathbb{P}^5$ is a hyperplane (that is a linear space defined by a single equation).*

Proof. Indeed, if $p = [u : v : w]$, then $[a : b : c : d : e : f] \in H_p$ iff $[a : b : c : d : e : f]$ satisfies the linear equation
$$u^2 A + v^2 B + w^2 C + (vw)D + (uw)E + (uv)F = 0. \qquad \square$$

For example, the conic passes through $p = [0 : 0 : 1]$ iff the coefficient of Z^2 is zero iff $c = 0$.

Lemma 2.4. *Suppose we are given five points p_1, p_2, p_3, p_4 and p_5, and we are working over an infinite field.*

Then, either there is a unique conic through these points, or infinitely many.

Proof. Let $H_i \subset \mathbb{P}^5$ be the hyperplane corresponding to p_i. Then the set of conics passing through the given points corresponds to the intersection of the five hyperplanes. As the intersection of linear spaces is a linear space, the result follows. $\qquad \square$

Definition 2.5. *Let $X \subset \mathbb{P}^n$. We say that X is in **linear general position** if every subset with $k < n + 2$ points spans a linear space of dimension $k - 1$.*

Here the span of points in $\mathbb{P}^n$ is defined as it is in $\mathbb{A}^n$. Thus a collection of points in the plane is in linear general position, if no three points are collinear, a collection of points in $\mathbb{P}^3$ is in linear general position, if no three points are collinear and no four points are coplanar, and so on.

Note also that if X has more than n points, we only have to check that no subset of $n+1$ points are contained in a hyperplane.

Theorem 2.6. *There is a unique conic passing through five points in linear general position.*

Proof. Suppose not. Then the intersection of the five hyperplanes H_1, H_2, H_3, H_4 and H_5 would contain a line, call it $l \subset \mathbb{P}^5$. Pick two points of this line, corresponding to two quadratic polynomials F and G. As any two points on l, span l, the general point of l is given as $[sF + tG]$, for $[s : t] \in \mathbb{P}^1$. Thus the curve $sF + tG = 0$ contains the five given points p_1, p_2, p_3, p_4 and p_5.

Pick any point $p \in \mathbb{P}^2$. Then we may find $[s : t] \in \mathbb{P}^1$ such that $(sF + tG)(p) = 0$. Indeed, if $G(p) = 0$, take $[s : t] = [0 : 1]$, else set $s = 1$ and
$$t = -\frac{F(p)}{G(p)}.$$
Now pick p collinear with p_1 and p_2. Then the conic C corresponding to $sF + tG = 0$ contains the three points p_1, p_2 and p of the line $m = \langle p_1, p_2 \rangle$. Pick coordinates so that m is given as $Z = 0$. Then the quadratic polynomial
$$F(X, Y, 0),$$
has three zeroes. It follows that $F(X, Y, 0) = 0$, so that $F(X, Y, Z) = ZG(X, Y, Z)$. In other words the curve C is the union of the two lines $Z = 0$ and $G = 0$. But then one of the two lines contains three of our five points, which contradicts our assumption that the points are in linear general position. $\qquad\square$

Corollary 2.7. *There is a unique circle passing through three non-collinear points in $\mathbb{R}^2$.*

Proof. Note that the line spanned by the points $[1 : \pm i : 0]$ is the line at infinity of $\mathbb{P}^2_{\mathbb{C}}$. Thus given three points p, q and r in $\mathbb{R}^2$, which are not collinear, then the five points p, q, r and $[1 : \pm i : 0]$ are in linear general position in $\mathbb{P}^2_{\mathbb{C}}$.

By (2.6) there is a unique conic through the five given points. Now of the five hyperplanes that define this conic, three are defined by linear equations with real coefficients and even though the other two have complex roots, the equations of the hyperplanes are complex conjugates. Since the set of solutions to a set of equations which is invariant under complex conjugation, is invariant under complex conjugation, it follows that this unique solution has coefficients which are invariant under complex conjugation, which is to say that it is a point with real coordinates. In particular the definining equation of the unique conic

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passing through the five given point is real. On the other hand the corresponding curve contains three real points. Therefore by (2.1) there is a unique circle through the three real points. □

Note the fancy footwork needed to deal with the problem of working over non algebraically closed fields.

It turns out there are other ways to prove (2.6). To give these other ways, we need to talk about maps between varieties.