17. Toric varieties

First some stuff about algebraic groups;

Definition 17.1. *Let G be a group. We say that G is an* ***algebraic group*** *if G is a quasi-projective variety and the two maps $m\colon G\times G \longrightarrow G$ and $i\colon G \longrightarrow G$, where m is multiplication and i is the inverse map, are both morphisms.*

It is easy to give examples of algebraic groups. For example, consider the group $G = \mathrm{GL}_n(K)$. In this case G is an open subset of $\mathbb{A}^{n^2}$, the complement of the determinant, which expands to a polynomial. Matrix multiplication is obviously a morphism, and the inverse map is a morphism by Cramer's rule. Note that there are then many obvious algebraic subgroups; the orthogonal groups, special linear group and so on. Clearly PGL_n is also an algebraic group; indeed the quotient of an algebraic group by a closed normal subgroup is an algebraic group. All of these are affine algebraic groups.

Definition 17.2. *Let G be an algebraic group. If G is affine then we say that G is a* ***linear algebraic group****. If G is projective and connected then we say that G is an* ***abelian variety****.*

Note that any finite group is an algebraic group (both affine and projective). Note that it turns out that any affine group is always a subgroup of a matrix group, so that the notation makes sense.

Definition 17.3. *The group $\mathbb{G}_m$ is $\mathrm{GL}_1(K)$. The group $\mathbb{G}_a$ is the subgroup of $\mathrm{GL}_2(K)$ of upper triangular matrices with one's on the diagonal.*

Note that as a group $\mathbb{G}_m$ is the set of units in K under multiplication and $\mathbb{G}_a$ is equal to K under addition, and that both groups are affine of dimension 1; in fact they are the only linear algebraic groups of dimension one, up to isomorpism. Note that if we are given a linear algebraic group G, we get a Hopf algebra A.

Indeed if A is the coordinate ring of G, then A is a K-algebra and there are maps

$$A \longrightarrow A \otimes A \qquad \text{and} \qquad A \longrightarrow A,$$

induced by the multiplication and inverse map for G.

Definition 17.4. *The algebaic group $\mathbb{G}_m^k$ is called a* ***torus****.*

So a torus in algebraic geometry is just a product of copies of $\mathbb{G}_m$.

In fact one can define what it means to be a group scheme:

Definition 17.5. *Let $\pi\colon X \longrightarrow S$ be a morphism of schemes. We say that X is a* ***group scheme*** *over S, if there are three morphisms, $e\colon S \longrightarrow X$, $\mu\colon X \underset{S}{\times} X \longrightarrow X$ and $i\colon X \longrightarrow X$ over S which satisfy the obvious axioms.*

We can define a group scheme $\mathbb{G}_{m,\operatorname{Spec}\mathbb{Z}}$ over $\operatorname{Spec}\mathbb{Z}$, by taking

$$\operatorname{Spec}\mathbb{Z}[x, x^{-1}].$$

Given any scheme, this gives us a group scheme $\mathbb{G}_{m,S}$ over S, by taking the fibre square. In the case when $S = \operatorname{Spec} k$, k an algebraically closed field, then $\mathbb{G}_{m,\operatorname{Spec} k}$ is $t(\mathbb{G}_m)$, the scheme associated to the quasi-projective variety $\mathbb{G}_m$. We will be somewhat sloppy and not be to careful to distinguish the two cases.

Similarly we can take

$$\mathbb{G}_{a,\operatorname{Spec}\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[x].$$

Definition 17.6. *Let G be an algebraic group and let X be a variety acted on by G, $\pi\colon G \times X \longrightarrow X$. We say that the action is* ***algebraic*** *if π is a morphism.*

For example the natural action of $\mathrm{PGL}_n(K)$ on $\mathbb{P}^n$ is algebraic, and all the natural actions of a group on itself are algebraic.

Definition 17.7. *We say that a quasi-projective variety X is a* ***toric variety*** *if there is a dense open subset U isomorphic to a torus, such that the natural action of U on itself extends to an action on the whole of X.*

For example, any torus is a toric variety. $\mathbb{A}^n_k$ is a toric variety. The natural torus is the complement of the coordinate hyperplanes and the natural action is as follows

$$((t_1, t_2, \dots, t_n), (a_1, a_2, \dots, a_n)) \longrightarrow (t_1 a_1, t_2 a_2, \dots, t_n a_n).$$

More generally, $\mathbb{P}^n$ is a toric variety. The action is just the natural action induced from the action above. A product of toric varieties is toric.

One thing to keep track of are the closures of the orbits. For the torus there is one orbit. For affine space and projective space the closure of the orbits are the coordinate subspaces.

Definition 17.8. *Let M be a lattice and let $N = \operatorname{Hom}(M, \mathbb{Z})$ be the dual lattice.*

A ***strongly convex rational polyhedral cone*** *$\sigma \subset N_{\mathbb{R}} = \otimes N.\mathbb{Z}/\mathbb{R}$. is the intersection of finitely many half spaces (polyhedral) which are* *defined by equations with rational coefficients (rational) which contains the origin (cone) but no other linear subspace (strongly convex).*

*A **fan** in N is a set F of finitely many strongly convex rational polyhedra, such that the intersection of any two cones in F is a cone in F and two cones intersect in a face of each.*

The basic toric varieties are in bijection with fans (up to the action of $\mathrm{SL}(n, \mathbb{Z})$.

Suppose we start with σ. Form the dual cone

$$\sigma^{\perp} = \{ v \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0, u \in \sigma \}.$$

Now take the integral points,

$$S_{\sigma} = \sigma^{\perp} \cap M.$$

Then form the group algebra,

$$A_{\sigma} = \mathbb{C}[S_{\sigma}].$$

Finally form the affine variety,

$$U_{\sigma} = \operatorname{Spec} A_{\sigma}.$$

For example, suppose we start with $M = \mathbb{Z}^2$, σ the cone spanned by $(1, 0)$ and $(0, 1)$. Then $\sigma^{\perp}$ is the same and the group algebra is $\mathbb{C}[x, y]$. So we get $\mathbb{A}^2$. Similarly if we start with the cone spanned by $e_1, e_2, \dots, e_n$ inside $N_{\mathbb{R}}$.

Now suppose we start with $\sigma = \{0\}$ in $\mathbb{R}$. Then $\sigma^{\perp}$ is the whole of M. $\mathcal{C}[M] = \mathbb{C}[x, x^{-1}]$ and taking spec we get a torus. More generally we always get a torus. Note that if $\tau \subset \sigma$ is a face then $\sigma^{\perp} \subset \tau^{\perp}$, so that $A_{\sigma} \subset A_{\tau}$. In fact A_{τ} is a localisation of A_{σ}, so that $U_{\tau} \subset U_{\sigma}$ is an open affine subset.