16. Prim and Proper

Okay prim is not a property of schemes, but proper and separated are. In this section we want to extend the intuitive notions of being Hausdorff and compact to the category of schemes.

First we come up with a formal definition of both properties and then we investigate how to check the formal definitions in practice. We start with the definition of separated, which should be thought as corresponding to Hausdorff.

Definition 16.1. *Let $f\colon X \longrightarrow Y$ be a morphism of schemes. The* ***diagonal morphism*** *is the unique morphism $\Delta\colon X \longrightarrow X \underset{Y}{\times} X$, given by applying the universal property of the fibre product to the identity map $X \longrightarrow X$, twice.*

We say that the morphism f is ***separated*** *if the diagonal morphism is a closed immersion.*

Example 16.2. *Consider the line X, with a double origin, obtained by gluing together two copies of $\mathbb{A}^1_k$, without identifying the origins. Consider the fibre square over k, $X \underset{k}{\times} X$. This is a doubled affine plane, which has two x-axes, two y-axes and four origins. The diagonal morphism, only hits two of those four origins, whilst the closure contains all four origins.*

Proposition 16.3. *Every morphism of affine schemes is separated.*

Proof. Suppose we are given a morphism of affine schemes $f\colon X \longrightarrow Y$, where $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. Then the diagonal morphism is given by,

$$A \underset{B}{\otimes} A \longrightarrow A \qquad \text{where} \qquad a \otimes a' \longrightarrow aa'.$$

As this is a surjective ring homomorphism, it follows that Δ is a closed immersion. $\square$

Corollary 16.4. *$f\colon X \longrightarrow Y$ is separated iff the image of the diagonal morphism is a closed subset.*

Proof. One direction is clear. So suppose that the image of the diagonal morphism is closed. We need to prove that $\Delta\colon X \longrightarrow \Delta(X)$ is a homeomorphism and that $\mathcal{O}_{X \underset{Y}{\times} X} \longrightarrow \Delta_* \mathcal{O}_X$ is surjective. Consider the first projection $p_1\colon X \underset{Y}{\times} X \longrightarrow X$. As the composition $p_1 \circ \Delta$ is the identity, it is immediate that Δ is a homeomorphism onto its image.

To check surjectivity of sheaves, we may work locally. Pick $p \in X$ and an open affine neighbourhood $V \subset Y$ of the image $q \in Y$. Let U

be an open affine neighbourhood of p contained in the inverse image of V. Then $U \underset{V}{\times} U$ is an open affine neighbourhood of $\Delta(p)$, and by (16.3), $\Delta\colon U \longrightarrow U \underset{V}{\times} U$ is a closed immersion. But then the map of sheaves is surjective on stalks at p. $\square$

The idea of how to characterise both properties (separated and proper), is based on the idea of probing with curves. After all, the clasic example of the doubled origin, admits an open immersion with two different extensions. Firstly, we need to work more locally than this, so that we want to work with local rings. However, our schemes are so general, that we also need to work with something more general than a curve. We need to work with spectrum of a valuation ring.

We will need some results from commutative algebra to do with valuation rings.

Definition 16.5. *Let A and B be two local rings, with the same field of fractions. We say that B* ***dominates*** *A if $A \subset B$ and $m_A = m_B \cap A$.*

Lemma 16.6. *Let R be a local ring which is an integral domain with field of fractions K. Then R is a valuation ring iff it is maximal with respect to dominance. Every local ring in K is contained a valuation ring.*

Definition 16.7. *Let X be a topological space. We say that x_0 is a* ***specialisation*** *of x_1 if $x_0 \in \overline{\{x_1\}}$.*

Lemma 16.8. *Let R be a valuation ring with quotient field K. Let $T = \operatorname{Spec} R$ and let $U = \operatorname{Spec} K$. Let X be any scheme.*

(1) *To give a morphism $U \longrightarrow X$ is equivalent to giving a point $x_1 \in X$ and an inclusion of fields $k(x_1) \subset K$.*
(2) *To give a morphism $T \longrightarrow X$ is equivalent to giving two points x_0, $x_1 \in X$, with x_0 a specialisation of x_1 and an inclusion of fields $k(x_1) \subset K$, such that R dominates the local ring $\mathcal{O}_{Z,x_0}$, in the subscheme $Z = \overline{\{x_0\}}$ of X, with its reduced induced structure.*

Proof. We have already seen (1). Let t_0 be the closed point of T and let t_1 be the generic point. If we are given a morphism $T \longrightarrow X$, then let x_i be the image of t_i. As T is reduced, we have a factorisation $T \longrightarrow Z$. Moreover $k(x_1)$ is the function field of Z, so that there is a morphism of local rings $\mathcal{O}_{Z,x_0} \longrightarrow R$ compatible with the inclusion $k(x_1) \subset K$. Thus R dominates $\mathcal{O}_{Z,x_0}$.

Conversely suppose given x_0 and x_1. The inclusion $\mathcal{O}_{Z,x_0} \longrightarrow R$ gives a morphism $T \longrightarrow \operatorname{Spec} \mathcal{O}_{Z,x_0}$, and composing this with the natural map $\operatorname{Spec} \mathcal{O}_{Z,x_0} \longrightarrow X$ gives a morphism $T \longrightarrow X$. $\square$

Lemma 16.9. *Let $f\colon X \longrightarrow Y$ be a compact morphism of schemes*

Then $f(X)$ is closed iff it is stable under specialisation.

Proof. Let us in addition suppose that f is of finite type and that X and Y are noetherian. Then $f(X)$ is constructible by Chevalley's Theorem, whence closed. For the general case, see Hartshorne, II.4.5. $\square$

Now we are ready to state:

Theorem 16.10 (Valuative Criterion of Separatedness)**.** *Let $f\colon X \longrightarrow Y$ be a morphism of schemes, and assume that X is Noetherian. Then f is separated iff the following condition holds:*

For any field K and for any valuation ring R with quotient field K, let $T = \operatorname{Spec} R$, let $U = \operatorname{Spec} K$ and let $i\colon U \longrightarrow T$ be the morphism induced by the inclusion $R \subset K$. Given morphisms $T \longrightarrow Y$ and $U \longrightarrow X$ which makes a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

there is at most one morphism $T \longrightarrow X$ which makes the diagram commute.

Proof. Suppose that f is separated, and that we are given two morphisms $h\colon T \longrightarrow X$ and $h'\colon T \longrightarrow X$, which make the diagram commute.

Then we get a morphism $h''\colon T \longrightarrow X \underset{Y}{\times} X$. As the restrictions of h and h' to U agree, it follows that h'' sends the generic point t_1 of T to a point of the diagonal $\Delta(X)$. Since the diagonal is closed, it follows that t_0 is sent to a point of the diagonal. But then the images of t_0 and t_1, under h and h', are the same points x_0 and $x_1 \in X$. Since the inclusion $k(x_1) \subset K$ comes out the same, it follows that $h = h'$.

Now let us prove the other direction. It suffices to prove that $\Delta(X)$ is closed in $X \underset{Y}{\times} X$, which in turn is equivalent to proving that it is stable under specialisation. Suppose that $\xi_1 \in \Delta(X)$ and suppose that ξ_0 is in the closure of $\{\xi_1\}$. Let $K = k(\xi_1)$ and let A be the local ring of ξ_0 in the closure of ξ_1. Then $A \subset K$ and so there is a valuation ring R which dominates A. Then by (16.8) there is a morphism $T \longrightarrow X \underset{Y}{\times} X$, where

$T = \operatorname{Spec} R$, sending t_i to ξ_i. Composing with either projection down to X, we get two morphisms $T \longrightarrow X$, which give the same morphism to Y and whose restrictions to U are the same, as $\xi_1 \in \Delta(X)$. By assumption then, these two morphisms agree, and so the morphism $T \longrightarrow X \underset{Y}{\times} X$ must factor through Δ. But then $\xi_0 \in \Delta(X)$, whence $\Delta(X)$ is closed. □

Corollary 16.11. *Assume that all schemes are Noetherian.*

(1) *Open and closed immersions are separated.*
(2) *A composition of separated morphisms is separated.*
(3) *Separated morphisms are stable under base change.*
(4) *If $f\colon X \longrightarrow Y$ and $f'\colon X' \longrightarrow Y'$ are two separated morphisms over a scheme S, then the product morphism $f \times f'\colon X \underset{S}{\times} X' \longrightarrow Y \underset{S}{\times} Y'$ is also separated.*
(5) *If $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ are two morphisms, such that $g \circ f$ is separated, then f is separated.*
(6) *A morphism $f\colon X \longrightarrow Y$ is separated iff Y can be covered by open subsets V_i such that $f^{-1}(V_i) \longrightarrow V_i$ is separated for each i.*

Proof. These all follow from (16.10). For example, consider the proof of (2). We are given $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$, two separated morphisms. By assumption we are given two morphisms $h\colon T \longrightarrow X$ and $h'\colon T \longrightarrow X$, as in (16.10). By composition with f, these give two morphisms $k\colon T \longrightarrow Y$ and $k'\colon T \longrightarrow Y$. As g is separated, these morphisms agree. But then as f is separated, $h = h'$. □

Now we turn to the notion of properness.

Definition 16.12. *A morphism $f\colon X \longrightarrow Y$ is **proper** if it is separated, of finite type, and universally closed.*

Example 16.13. *The affine line $\mathbb{A}^1_k$ is certainly separated and of finite type over k. However it is not proper over k, since it is not universally closed. Indeed consider $\mathbb{A}^2_k = \mathbb{A}^1_k \underset{k}{\times} \mathbb{A}^1_k$. The image of the hyperbola under either of the projection maps is not closed.*

Theorem 16.14 (Valuative Criterion of Properness)**.** *Let $f\colon X \longrightarrow Y$ be a morphism of finite type, with X Noetherian. Then f is proper iff for every valuation ring R and for every pair of morphisms $U \longrightarrow Y$*

and $T \longrightarrow Y$ which form a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

there is a unique morphism $h\colon T \longrightarrow X$ making the diagram commute.

Proof. Suppose that f is proper. Then f is certainly separated, so h, if it exists, is surely unique. Consider the base change given by $T \longrightarrow Y$, and set $X_T = X \underset{Y}{\times} T$. We get a morphism $U \longrightarrow X_T$, applying the universal property to the morphisms $U \longrightarrow X$ and $U \longrightarrow T$.

$$\begin{tikzcd} U \arrow[r] & X_T \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"'] \\ & T \arrow[r] & Y. \end{tikzcd}$$

Let $\xi_1 \in X_T$ be the image of the point $t_1 \in U$. Let $Z = \overline{\{\xi_1\}}$. As f is proper, f' is closed and so $f'(Z) \subset T$ is closed. Thus $f'(Z) = T$, as $f'(Z)$ contains the generic point of T. Pick $\xi_0 \in Z$ such that $f(\xi_0) = t_0$. Then we get a morphism of local rings $R \longrightarrow \mathcal{O}_{Z,\xi_0}$. Now the function field of Z is $k(\xi_1)$ which by construction is a subfield of K. Now R is maximal with respect to dominance in K. Thus $R \simeq \mathcal{O}_{Z,\xi_0}$. Thus by (16.8) there is a morphism $T \longrightarrow X_T$ sending t_i to ξ_i. Now compose with the natural map $X_T \longrightarrow X$.

Now suppose that f satisfies the given condition. Let $Y' \longrightarrow Y$ be an arbitrary base change, and let $X' \longrightarrow X$ be the induced morphism. Pick a closed subset $Z \subset X'$, imbued with the reduced induced structure:

$$\begin{tikzcd} Z & \subset & X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"'] \\ & & Y' \arrow[r] & Y. \end{tikzcd}$$

We want to prove that $f(Z)$ is closed. Now f is of finite type by assumption, so that f' is of finite type. It suffices to show that $f(Z)$ is closed under specialisation, by (16.9).

Pick a point $z_1 \in Z_1$ and let $y_1 = f(z_1)$. Suppose that y_0 is in the closure of y_1. Let S be the local ring of the closure of y_1 at y_0. Then the quotient field of S is $k(y_1)$ which is a subfield of $K = k(z_1)$. Pick a valuation ring R contained in K which dominates S. Then by (16.8), we get a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & Z \arrow[d] \\ T \arrow[r] & Y'. \end{tikzcd}$$

Composing with the morphisms $Z \longrightarrow X' \longrightarrow X$ and $Y' \longrightarrow Y$ we get morphisms $U \longrightarrow X$ and $T \longrightarrow Y$. By hypothesis, there is a morphism $T \longrightarrow X$ which makes the diagram commute. By the universal property of a fibre product, this lifts to a morphism $T \longrightarrow X$. As Z is closed, this factors into $T \longrightarrow Z$. Let z_0 be the image of t_0. Then z_0 maps to y_0, as required. □

Corollary 16.15. *Assume that all schemes are Noetherian.*

(1) *A closed immersion is proper.*
(2) *A composition of proper morphisms is proper.*
(3) *Proper morphisms are stable under base change.*
(4) *If $f \colon X \longrightarrow Y$ and $f' \colon X' \longrightarrow Y'$ are two proper morphisms over a scheme S, then the product morphism $f \times f' \colon X \underset{S}{\times} X' \longrightarrow Y \underset{S}{\times} Y'$ is also proper.*
(5) *If $f \colon X \longrightarrow Y$ and $g \colon Y \longrightarrow Z$ are two morphisms, such that $g \circ f$ is proper and g is separated, then f is proper.*
(6) *A morphism $f \colon X \longrightarrow Y$ is proper iff Y can be covered by open subsets V_i such that $f^{-1}(V_i) \longrightarrow V_i$ is proper for each i.*

Proof. These all follow from (16.14). □

Definition 16.16. *Let $f \colon X \longrightarrow Y$ be a morphism of schemes. We say that f is projective if it is the composition of a closed immersion $X \longrightarrow \mathbb{P}^n_Y$ and the natural projection $\mathbb{P}^n_Y \longrightarrow Y$.*

Theorem 16.17. *A projective morphism is proper.*

Proof. Since a closed immersion is of finite type, and using the results of (16.11) and (16.15), it suffices to prove that $X = \mathbb{P}^n_{\operatorname{Spec}\mathbb{Z}}$ is proper over $\operatorname{Spec}\mathbb{Z}$. Now X is covered by open affines of the form $U_i = \operatorname{Spec}\mathbb{Z}[x_1, x_2, \dots, x_n]$. Thus X is certainly of finite type over $\operatorname{Spec}\mathbb{Z}$. We check the condition of (16.14). Suppose we have a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & \operatorname{Spec}\mathbb{Z}. \end{tikzcd}$$

Let $\xi_1 \in X$ be the image of the unique point of U. By induction on n, we may assume that ξ_1 is not contained in any of the $n+1$ standard hyperplanes, so that $\xi_1 \in U = \bigcap U_i$. Thus the functions x_i/x_j are all invertible on U.

There is an inclusion $k(\xi_1) \subset K$. Let f_{ij} be the image of x_i/x_j. Then

$$f_{ik} = f_{ij} f_{jk}.$$

Let $\nu \colon K \longrightarrow G$ be the valuation associated to R. Let $g_i = \nu(f_{i0})$ and pick k such that g_k is minimal. Then

$$\nu(f_{ik}) = g_i - g_k \geq 0.$$

Hence $f_{ik} \in R$. Define a ring homomorphism

$$\mathbb{Z}[x_0/x_k, x_1/x_k, \dots, x_n/x_k] \longrightarrow R \qquad \text{by sending} \qquad x_i/x_k \longrightarrow f_{ik}.$$

This gives a homomorphism $T \longrightarrow V_k$ and by composition a morphism $T \longrightarrow X$. □

Using this, we can finally characterise the image of the functor t. We will temporarily adopt the condition that every variety is irreducible.

Proposition 16.18. *Fix an algebraically closed field k. Then the image of the functor t is precisely the set of integral quasi-projective schemes, and the image of a projective variety is an integral projective scheme.*

In particular for every variety V, $t(V)$ is an integral separated scheme of finite type over k.

Proof. It only suffices to prove that every integral projective scheme Y is the image of a variety V. Let Y be a closed subscheme of $\mathbb{P}^n_k$. Then the set of closed points is a closed subset V of the variety $\mathbb{P}^n$. If Y is irreducible then V is irreducible, as V is dense in Y. If Y is reduced, it is easy to see that $t(V) = Y$, since they have the same support and they are both reduced. □

Definition 16.19. *A **variety** V is an integral separated scheme of finite type over an algebraically closed field. If in addition V is proper, then we say that V is a **complete variety**.*

We will see examples of complete varieties that are not projective.