15. Images of varieties

This section is centred around anwering the following natural:

Question 15.1. *Let $f\colon X \longrightarrow Y$ be a morphism of quasi-projective varieties. If Z is a closed subset then what can we say about the image $f(Z)$?*

Definition 15.2. *Let $f\colon X \longrightarrow Y$ be a continuous map of topological spaces. We say that f is* ***closed*** *if the image of a closed set is closed.*

We say that a morphism of quasi-projective varieties is ***proper*** *if it is closed.*

Our first guess is that morphisms are proper. Unfortunately this is not correct. Let $X = \mathbb{A}^2$, $Y = \mathbb{A}^1$, f projection onto the first factor. Let

$$Z = \{\, (x, y) \in \mathbb{A}^2 \mid xy = 1 \,\}.$$

Then the image of Z is $\mathbb{A}^1 - \{0\}$, an open subset of $\mathbb{A}^1$, not a closed subset.

Okay, well perhaps the image of a quasi-projective variety is quasi-projective (perhaps even the image of affine is affine?)? Unfortunately this is not true either. For example, take $X = Y = Z = \mathbb{A}^2$. Let $f\colon X \longrightarrow Y$ be the morphism $(a, b) \longrightarrow (a, ab)$. Let us determine the image I. Pick $(x, y) \in \mathbb{A}^2$. If $x \neq 0$, then take $a = x$ and $b = y/x$. Then $(a, ab) = (x, y)$. Thus I contains the complement of the x-axis. Now if $y \neq 0$ and $x = 0$, then (x, y) is surely not in the image. However $(0, 0)$ is in the image; indeed it is the image of $(0, 0)$. Thus the image is equal to the complement of the x-axis union the origin, which is not locally closed.

In fact, it turns out that this is as bad as it gets.

Definition 15.3. *Let X be a topological space. A subset $Z \subset X$ is said to be* ***constructible*** *if it is the finite union of locally closed subsets.*

The image I of $\mathbb{A}^2$ above is the union of the open set U_x and the closed set $(0, 0)$, so I is constructible. In fact, we are aiming for:

Theorem 15.4 (Chevalley's Theorem)**.** *Let $\pi\colon X \longrightarrow Y$ be morphism of quasi-projective varieties.*

Then the image of a constructible set is constructible.

The first case to deal with, in fact the crucial case, which is of considerable interest in its own right, is the case when X is projective:

Theorem 15.5. *If $f\colon X \longrightarrow Y$ is a morphism of quasi-projective varieties and X is projective then f is proper.*

It turns out that it is easier to prove a stronger result than (15.5).

Definition 15.6. *Let $i\colon X \longrightarrow Y$ be a morphism of quasi-projective varieties. We say that i is a* ***closed immersion*** *if the image of i is closed and i is an isomorphism onto its image.*

Definition 15.7. *Let $\pi\colon X \longrightarrow Y$ be a morphism of quasi-projective varieties.*

We say that π is a ***projective morphism*** *if it can be factored into a closed immersion $i\colon X \longrightarrow \mathbb{P}^n \times Y$ and the projection morphism $\mathbb{P}^n \times Y \longrightarrow Y$.*

Lemma 15.8. *Every morphism from a projective variety is projective.*

Proof. Just take the graph. □

Thus to prove (15.5) it suffices to prove:

Theorem 15.9. *Every projective morphism is proper.*

Clearly closed embeddings are proper and the composition of proper maps is proper. So to prove (15.9) we need to prove:

Lemma 15.10. *If $\pi\colon \mathbb{P}^n \times Y \longrightarrow Y$ denotes projection onto the second factor then π is proper.*

The trick is to reduce to the case $n = 1$. The idea is that projective space $\mathbb{P}^n$, via projection, is very close to the product $\mathbb{P}^1 \times \mathbb{P}^{n-1}$.

Definition-Lemma 15.11. *Let $\Lambda \subset \mathbb{P}^n$ be a linear subspace and let $\Lambda' = \mathbb{P}^k$ be a complimentary linear subspace. The map*

$$\pi = \pi_\Lambda\colon \mathbb{P}^n - \Lambda \longrightarrow \mathbb{P}^k,$$

given by sending p to the unique point $q = \pi(p) = \langle \Lambda, p\rangle \cap \Lambda'$ is a morphism, which is called ***projection from*** *Λ.*

Proof. We only need to check that π is a morphism. This is easy however you do it, but it is particularly easy if we choose coordinates so that Λ is given by the vanishing of the last $n-k$-coordinates. In this case case π is given by the map

$$[X_0 : X_1 : \dots : X_n] \longrightarrow [X_0 : X_1 : \dots : X_k].$$ □

Definition 15.12. *Let $\pi\colon X \longrightarrow Y$ be a morphism of quasi-projective varieties. We say that π is a* ***fibre bundle****, with fibre F, if we can find a cover of the base Y, such that over each open subset U of the cover, $\pi^{-1}(U)$ is isomorphic to $U \times F$, over U.*

Note that if π is a fibre bundle then every fibre of π is surely a copy of F. It is convenient to denote $\pi^{-1}(U)$ by $X|_U$.

Lemma 15.13. *The graph of the projection map from a point p defines a morphism $\Gamma \longrightarrow \mathbb{P}^{n-1}$, which is a fibre bundle, with fibre $\mathbb{P}^1$.*

Proof. The rational map given by projection from a point p

$$\pi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1},$$

is clearly defined everywhere, except at the point p of projection. Moreover this map is clearly constant on any line through p. Thus the morphism $\Gamma \longrightarrow \mathbb{P}^{n-1}$ has fibres equal to the lines through p.

Pick two hyperplanes H_1 and H_2, neither of which contain p. Under projection, we may indentify H_1 with the base $\mathbb{P}^{n-1}$. Let $V = H_1 \cap H_2$. Then the image of V is a hyperplane in $\mathbb{P}^{n-1}$. Let U be the complement of this hyperplane in $\mathbb{P}^{n-1}$. Projection from V defines a rational map down to $\mathbb{P}^1$,

$$\pi_V\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^1.$$

This rational map is an isomorphism on every line l through p which does not intersect V.

Define a morphism $\psi\colon \Gamma|_U \longrightarrow \mathbb{P}^1 \times U$ via these two projection maps. Let $\phi\colon \mathbb{P}^1 \times U \longrightarrow \Gamma|_U$ be the map defined by sending (p_1, p_2) to the intersection of $\langle \Lambda, p_1 \rangle$ and $\langle p, p_2 \rangle$. Then ϕ is a morphism and it is not hard to see that ϕ is the inverse of ψ. Thus ψ is an isomorphism. Fixing H_1 and varying H_2 it is clear that we get a cover of $\mathbb{P}^{n-1}$ in this way. $\square$

Lemma 15.14. *To prove* (15.10) *we may assume that $n = 1$.*

Proof. Let $X \subset \mathbb{P}^n \times Y$ be a closed subset and let I be the image of X under projection down to Y. If we set

$$Z = \{\, y \in Y \mid \mathbb{P}^n \times \{y\} \subset X \,\},$$

then Z is closed and of course $Z \subset I$. So it suffices to prove that $I \cap (Y - Z)$ is closed. Replacing Y by $Y - Z$ we may as well assume that Z is empty.

Pick $y \in Y$. As we are assuming that Z is empty we may find $p \in \mathbb{P}^n$ such that $(p, y) \notin X$. If we set

$$U = \{\, y \in Y \mid (p, y) \notin X \,\},$$

then U is an open subset of Y. As the problem of showing I is closed is local on Y, replacing Y by U, we may assume that $U = Y$.

Let $q\colon \Gamma \times Y \longrightarrow \mathbb{P}^n \times Y$ the morphism which is the identity on Y and the graph of the blow up on $\mathbb{P}^n$. Let X' be the strict transform of X. By definition X' is closed. On the other hand, by assumption X does not intersect $\{p\} \times Y$ so that X' is equal to the total transform of X. In particular the images of X' and X in Y coincide.

The morphism $Y \times \Gamma \longrightarrow Y$ factors through $Y \times \mathbb{P}^{n-1}$. So, by induction on n, it suffices to prove that the image of X' in $Y \times \Gamma \longrightarrow Y \times \mathbb{P}^{n-1}$ is closed. But now we are done, as we can check this locally on $Y \times \mathbb{P}^{n-1}$ and by (15.13), Γ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^{n-1}$, locally over $\mathbb{P}^{n-1}$. $\square$

The idea now is to work locally on Y, so that Y is affine, and think of $Y \times \mathbb{P}^1$ as being $\mathbb{P}^1$ over the coordinate ring of Y.

Lemma 15.15. *Let Y be an affine variety and let $X \subset Y \times \mathbb{P}^1$ be a closed subset.*

Then X is defined by polynomials $F(S,T) \in A(Y)[S,T]$, where $[S : T]$ are homogeneous coordinates on $\mathbb{P}^1$.

Proof. We may assume that X is irreducible and $X_0 = X \cap (Y \times U_0)$ is dense in X. As Y is affine, then $Y \times U_0$ is affine and X_0 is defined by polynomials $f(s)$, where $s = S/T$ and the coefficients of f belong to $A(Y)$. If $F(S,T)$ is the homogenisation of f then $F(S,T) \in A(Y)[S,T]$ vanishes on X and the set of all such polynomials cuts out X set theoretically. $\square$

Given $y \in Y$ and $F(S,T) \in A(Y)[S,T]$, let $F_y = F_y(S,T) \in K[S,T]$ be the polynomial we obtain by subsituting in $y \in Y$ to the coefficients.

Lemma 15.16. *Let $X \subset Y \times \mathbb{P}^1$.*

Then $y \in \pi(X)$ iff for every pair of functions $F(S,T)$ and $G(S,T) \in A(Y)[S,T]$ vanishing on X, both $F_y(S,T)$ and $G_y(S,T)$ have a common zero on $\{y\} \times \mathbb{P}^1$.

Proof. One inclusion is clear. So suppose that $y \notin \pi(X)$. Pick $F(S,T)$ that does not vanish on $\{y\} \times \mathbb{P}^1$. Then $F_y(S,T)$ has only finitely many zeroes. For each such zero p_i, we may find $G^i(S,T)$ such that $G^i_y(S,T)$ does not vanish at p_i. Taking an appropriate linear combination of the G^i gives us a polynomial G such that F_y and G_y do not have a common zero. $\square$

Lemma 15.17. *To prove* (15.9) *we may assume that X is defined by two polynomials F and G.*

To finish off, the idea is to use elimination theory.

Definition-Lemma 15.18. *Let S be a ring, and let F and G be two polynomials in $S[X,Y]$, of degrees d and e.*

Let $R(F,G) \in S$ be the determinant of the following $(d+e) \times (d+e)$ square matrix

$$\begin{vmatrix} f_0 & f_1 & f_2 & \dots & f_{d-1} & f_d & \dots & 0 & \dots \\ f_0 & f_1 & f_2 & \dots & f_{d-1} & f_d & \dots & 0 & \dots \\ 0 & f_0 & f_1 & f_2 & \dots & f_{d-1} & f_d & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & f_0 & f_1 & f_2 & \dots & f_{d-1} & f_d \\ g_0 & g_1 & g_2 & \dots & g_{e-1} & g_e & \dots & 0 & \dots \\ g_0 & g_1 & g_2 & \dots & g_{e-1} & g_e & \dots & 0 & \dots \\ 0 & g_0 & g_1 & g_2 & \dots & g_{e-1} & g_e & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & g_0 & g_1 & g_2 & \dots & g_{e-1} & g_e \end{vmatrix},$$

where $f_1, f_2, \dots, f_d$ and $g_1, g_2, \dots, g_e$ are the coefficients of F and G.

Then for every maximal ideal m of S, $\bar{R}(F,G) = 0$ in the quotient ring S/m iff the two polynomials $\bar{F}$ and $\bar{G}$ have a common zero.

Proof. Since expanding a determinant commutes with passing to the quotient S/m, we might as well assume that $S = K$ is a field.

Now note that the rows of this matrix correspond to the polynomials $X^iY^{e-1-i}F$ and $X^jY^{d-1-j}G$, where $0 \leq i \leq e-1$ and $0 \leq j \leq d-1$, expanded in the standard basis of the vector space P_{d+e-1} of polynomials of degree $d+e-1$. Thus the determinant is zero iff the polynomials $\mathcal{B} = \{X^iY^{e-1-i}F, X^jY^{d-1-j}G\}$ are dependent, inside P_{d+e-1}.

To finish off then it suffices to prove that this happens only when the two polynomials share a common zero. Now note that P_{d+e-1} has dimension $d+e$. Thus the $d+e$ polynomials $\mathcal{B}$ are independent iff they are a basis. Suppose that they share a common zero. Then the space spanned by $\mathcal{B}$ is contained in the vector subspace of all polynomials vanishing at the given point, and so $\mathcal{B}$ does not span. Now suppose that they are dependent. Collecting terms, there are then two polynomials A and B of degrees $e-1$ and $d-1$ such that

$$AF + BG = 0.$$

Suppose that $d \leq e$. Then every zero of G must be a zero of AF. As G has e zeroes and A has at most $d-1$ zeroes, it follows that one zero of G must be a zero of F. □

Proof of (15.9). By (15.17) it suffices to prove the result when $n = 1$ and X is defined by two polynomials F and G. In this case $\pi(X)$ is precisely given by the resultant of F and G, which is an element of $A(Y)$. □

(15.5) has the following very striking consequence.

Corollary 15.19. *Every regular function on a connected projective variety is constant.*

Proof. By definition a regular function is a morphism $f\colon X \longrightarrow \mathbb{A}^1$. Now by (15.5) the image of X is closed in $\mathbb{A}^1$. The only closed subsets of $\mathbb{A}^1$ are finite sets of points or the whole of $\mathbb{A}^1$. On the other hand f extends in an obvious way to a morphism $g\colon X \longrightarrow \mathbb{P}^1$. We haven't changed the image, but the image is now also a closed subset of $\mathbb{P}^1$. Thus the image cannot be $\mathbb{A}^1$.

Thus the image is a finite set of points. As X is connected, the image is connected and so the image is a point. □

Corollary 15.20. *Let X be a closed and connected subset of an affine variety.*

If X is also projective then X is a point.

Proof. By assumption $X \subset \mathbb{A}^n$. Suppose that X contains at least two points. Then at least one coordinate must be different. Let f be the function on $\mathbb{A}^n$ corresponding to this coordinate. Then f restricts to a non-constant regular function on X, which contradicts (15.5). □

Corollary 15.21. *Let $X \subset \mathbb{P}^n$ be a closed subset and let H be a hypersurface.*

If X is not a finite set of points, then $H \cap X$ is non-empty.

Proof. Suppose not. We may assume that X is irreducible. Let G be the defining equation of H. Pick F of degree equal to the degree of G. Then F/G is a regular function on X, since G is nowhere zero on X, and we can choose F so that F/G is not constant. But this contradicts (15.19). □

We can now answer our original question. Note that constructible sets are closed under complements and finite intersections and unions.

Lemma 15.22. *Let X be a Noetherian topological space and let Z be a subset.*

Then Z is constructible iff it is of the form $Z = Z_0 - (Z_1 - (Z_2 - \cdots - Z_k))$, where Z_i are closed and decreasing subsets.

Proof. Suppose that Z is constructible. Let Z_0 be the closure of Z. Then Z is dense in Z_0 and as Z is constructible, Z contains an open dense subset of Z_0. Clearly the difference $Z_0 - Z$ is constructible. Let Z_1 be the closure. Then $Z \supset Z_0 - Z_1$. If $Z_0 = Z_1$ then since Z contains an open dense subset of Z_0, it follows that Z_0 is empty. Continuing in this way, we construct a decreasing sequence of closed subsets,

$$Z_0 \supset Z_1 \supset \cdots \supset Z_k \supset \dots.$$

As X is Noetherian this sequence must terminate.

Now suppose that Z is an alternating difference of closed subsets,

$$Z = Z_0 - (Z_1 - (Z_2 - \cdots - Z_{2k-1})).$$

Then $Z = (Z_0 - Z_1) \cup (Z_1 - Z_2) \cdots \cup (Z_{2k-2} - Z_{2k-1})$. $\square$

Proof of (15.4). As the image of a union is the union of the images, it suffices to prove that the image of a locally closed subset is constructible. Suppose that Z is a locally closed subset. Replacing X by the closure of Z and Y by the closure of the image, we may assume that $\pi|_Z$ is dominant. Suppose that $\pi(Z)$ contains an open subset. Replacing X by the complement of the inverse image, we are then done by Noetherian induction.

Thus we are reduced to proving that $\pi(Z)$ contains an open subset. Replacing X by an open subset, we may assume that X is affine. Replacing X by its graph and applying induction on n, we may assume that $X \subset \mathbb{A}^n$ and that the map is the restriction of the projection map

$$\mathbb{A}^n \longrightarrow \mathbb{A}^{n-1},$$

where

$$(x_1, x_2, \dots, x_n) \longrightarrow (x_1, x_2, \dots, x_{n-1}),$$

so that there is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\pi"'] & \mathbb{A}^n \arrow[d] \\ Y \arrow[r] & \mathbb{A}^{n-1}. \end{tikzcd}$$

Thus we may assume that $X \subset Y \times \mathbb{A}^1$ and that we are projecting onto Y. Clearly we may replace $\mathbb{A}^1$ by $\mathbb{P}^1$. As Y is affine, every closed subset of $Y \times \mathbb{P}^1$ is defined by polynomials $F(S,T) \in A(Y)[S,T]$.

Let V be the complement of Z in X, so that $Z = X - V$ and both X and V are closed in $\mathbb{A}^n$. Pick $G(S,T) \in A(Y)[S,T]$ vanishing on V but not on X.

Suppose that $X = Y \times \mathbb{P}^1$. Let

$$W = \{\, y \in Y \mid \{y\} \times \mathbb{P}^1 \subset V \,\},$$

If $y \in W$ then every coefficient of G_y vanishes. In particular W is contained in a proper closed subset of Y (the vanishing of the coefficients of G) and $\pi(Z)$ contains the the complement of this closed subset.

So we may assume that X is a proper closed subset of $Y \times \mathbb{P}^1$. Pick $F(S,T) \in A(Y)[S,T]$ vanishing on X. Since X is closed, $\pi(X)$ is closed, whence $\pi(X) = Y$. But $R(F,G)$ is a non-zero polynomial that vanishes on $\pi(V)$ and $\pi(Z)$ contains $U_{R(F,G)}$. $\square$