14. Rational maps

It is often the case that we are given a variety X and a morphism defined on an open subset U of X. As open sets in the Zariski topology are very large, it is natural to view this as a map on the whole of X, which is not everywhere defined.

Definition 14.1. *A* ***rational map*** $\phi\colon X \dashrightarrow Y$ *between quasi-projective varieties is a pair* (f, U) *where* U *is a dense open subset of* X *and* $f\colon U \longrightarrow Y$ *is a morphism of varieties. Two rational maps* (f_1, U_1) *and* (f_2, U_2) *are considered equal if there is a dense open subset* $V \subset U_1 \cap U_2$ *such that the two functions* $f_1|_V$ *and* $f_2|_V$ *are equal.*

It is customary to avoid using the pair notation and to leave U unspecified. We often say in this case that ϕ is defined on U. Note that if U and V are two dense open sets, and (f, U), (g, V) represent the same rational map, then $(h, U \cup V)$ also represents the same map, where h is defined in the obvious way. By Noetherian induction, it follows that there is a largest open set on which ϕ is defined, which is called the **domain of** ϕ. The complement of the domain is called the **locus of indeterminancy**.

One way to get a picture of a rational map, is to consider the graph.

Definition 14.2. *Let* $\phi\colon X \dashrightarrow Y$ *be a rational map.*

The ***graph of*** ϕ *is the closure of the graph of* f*, where the pair* (f, U) *represents* ϕ*.*

The ***image of*** ϕ *is the image of the graph of* ϕ *under the second projection.*

Note that the domain of ϕ is precisely the locus where the first projection map is an isomorphism.

Definition 14.3. *Let* $\phi\colon X \dashrightarrow Y$ *and* $\psi\colon Y \dashrightarrow Z$ *be two rational maps. Suppose that* $\phi = (f, U)$ *and* $\psi = (g, V)$ *and that* $f(U) \cap V$ *is non-empty. Then we may define the composition of* ϕ *and* ψ *by taking the pair* $(g \circ f, f^{-1}(V))$*.*

Note that in general, we cannot compose rational maps. The problem might be that the image of the first map might lie in the locus where the second map is not defined. However there will never be a problem if X is irreducible and ϕ is dominant:

Definition 14.4. *We say that* ϕ *is* ***dominant*** *if the closure of the image of* ϕ *is the whole of* Y*.*

Note that this gives us a category, the category of irreducible varieties and dominant rational maps.

Definition 14.5. *We say that a dominant rational map $\phi\colon X \dashrightarrow Y$ of irreducible quasi-projective varieties is birational if it has an inverse. In this case we say that X and Y are* ***birational****. We say that X is* ***rational*** *if it is birational to $\mathbb{P}^n$.*

It is interesting to see an example. Let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the map

$$[X : Y : Z] \longrightarrow [YZ : XZ : XY].$$

This map is clearly a rational map. It is called a **Cremona transformation**. Note that it is a priori not defined at those points where two coordinates vanish. To get a better understanding of this map, it is convenient to rewrite it as

$$[X : Y : Z] \longrightarrow [1/X : 1/Y : 1/Z].$$

Written as such it is clear that this map is an involution, so that it is in particular a birational map.

It is interesting to check whether or not this map really is well defined at the points $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$. To do this, we need to look at the graph.

Consider the following map,

$$\mathbb{A}^2 \dashrightarrow \mathbb{A}^1,$$

which assigns to a point $p \in \mathbb{A}^2$ the slope of the line connecting the point p to the origin,

$$(x, y) \longrightarrow x/y.$$

Now this map is not defined along the locus where $y = 0$. Replacing $\mathbb{A}^1$ with $\mathbb{P}^1$ we get a map

$$(x, y) \longrightarrow [x : y].$$

Now the only point where this map is not defined is the origin. We consider the graph,

$$\Gamma \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

Consider how the graph sits over $\mathbb{A}^2$. Outside the origin, the first projection is an isomorphism. Over the origin, the graph is contained a copy of the image, that is $\mathbb{P}^1$. Consider any line $y = tx$, through the origin. Then this line, minus the origin, is sent to the point with slope t. It follows that the closure of this line is sent to the point with slope t. Varying t, it follows that any point of the fibre over $\mathbb{P}^1$ is a point of the graph.

Thus the morphism $p\colon \Gamma \longrightarrow \mathbb{A}^2$ is an isomorphism outside the origin and contracts a whole copy of $\mathbb{P}^1$ to a point. For this reason, we call p a blow up.

Definition 14.6. *Let $\phi\colon X \dashrightarrow Y$ be a rational map, which is given locally by $f_1, f_2, \dots, f_k$. Let I be the ideal spanned by $f_1, f_2, \dots, f_k$. The induced morphism $p\colon \Gamma \longrightarrow X$ is called the* ***blow up of the ideal*** *I.*

Clearly p is always birational, as it is an isomorphism outside $V(I)$.

In our case $I = \langle x, y \rangle$, the maximal ideal of p, so that we call p the blow up of a point. Suppose we have coordinates $[S : T]$ on $\mathbb{P}^1$. Then outside of the origin, the graph satisfies the equation $xT = yS$. Thus the closure must satisfy the same equation. Since this equation determines the graph outside the origin, in fact the graph is defined by this equation (as the whole fibre over the origin lives in the graph, we don't need anymore equations).

The inverse image of the origin is called the **exceptional divisor**.

Definition 14.7. *Let $\pi\colon X \longrightarrow Y$ be a birational morphism. The locus where π is not an isomorphism is called the* ***exceptional locus****. If $V \subset Y$, the inverse image of V is called the* ***total transform****. Let Z be the image of the exceptional locus. Suppose that V is not contained in Z. The* ***strict transform of*** *V is the closure of the inverse image of $V - Z$.*

It is interesting to compute the strict transform of some planar curves. We have already seen that lines through the origin lift to curves that sweep out the exceptional divisor. In fact the blow up separates the lines through the origin. These are then the fibres of the second morphism.

Let us now take a nodal cubic,

$$y^2 = x^2 + x^3.$$

We want to figure out its strict transform, so that we need the inverse image in the blow up. Outside the origin, there are two equations to be satisfied,

$$y^2 = x^2 + x^3 \qquad \text{and} \qquad xT = yS.$$

Passing to the coordinate patch $y = xt$, where $t = T/S$, and substituting for y in the first equation we get

$$x^2t^2 - x^2 - x^3 = x^2(t^2 - x - 1).$$

Now if $x = 0$, then $y = 0$, so that in fact locally $x = 0$ is the equation of the exceptional divisor. So the first factor just corresponds to the exceptional divisor. The second factor will tell us what the closure of our curve looks like, that is the strict transform. Now over the origin, $x = 0$, so that $t^2 = 1$ and $t = \pm 1$. Thus our curve lifts to a curve which intersects the exceptional divisor in two points. (If we compute in the coordinate patch $x = sy$, we will see that the curve does not meet the point at infinity). These two points correspond to the fact that the nodal cubic has two tangent lines at the origin, one of slope 1 and one of slope -1. We call the closure of the inverse image outside the origin as the strict transform (the total transform being just the whole inverse image).

Now consider what happens for the cuspidal cubic, $y^2 = x^3$. In this case we get

$$(xt)^2 - x^3 = x^2(t^2 - x).$$

Once again the factor of x^2 corresponds to the fact that the inverse image surely contains the exceptional divisor. But now we get the equation $t^2 = 0$, so that there is only one point over the origin, as one might expect from the geometry.

Let us go back to the Cremona transformation. To compute what gets blown up and blown down, it suffices to figure out what gets blown down, by symmetry. Consider the line $X = 0$. If $bc \neq 0$, the point $[0 : b : c]$ gets mapped to $[0 : 0 : 1]$. Thus the strict transform of the line $X = 0$ in the graph gets blown down to a point. By symmetry the strict transforms of the other two lines are also blown down to points. Outside of the union of these three lines, the map is clearly an isomorphism.

Thus the involution blows up the three points $[0 : 0 : 1]$, $[0 : 1 : 0]$, and $[1 : 0 : 0]$ and then blows down the three disjoint lines. Note that the three exceptional divisors become the three new coordinate lines.

One of the most impressive results of the nineteenth century is the following characterisation of the birational automorphism group of $\mathbb{P}^2$.

Theorem 14.8 (Noether)**.** *The birational automorphism group is generated by a Cremona transformation and* PGL(3)*.*

This result is very deceptive, since it is known that the birational automorphism group is, by any standards, very large.

Definition 14.9. *A **rational function** is a rational map to $\mathbb{A}^1$.*

*The set of all rational functions, denoted $K(X)$, is called the **function field**.*

Lemma 14.10. *Let X be an irreducible variety.*

Then the function field is a field. If $U \subset X$ is any open affine subset, then function field is precisely the field of fractions of the coordinate ring.

Proof. Clear, since on an irreducible variety, any rational function is determined by its restriction to any open subset, and locally any morphism is given by a rational function. $\square$

Proposition 14.11. *Let K be an algebraically closed field.*

Then there is an equivalence of categories between the category of irreducible varieties over K with morphisms the dominant rational maps, and the category of finitely generated field extensions of K.

Proof. Define a functor F from the category of varieties to the category of fields as follows. Given a variety X, let $K(X)$ be the function field of X. Given a rational map $\phi\colon X \dashrightarrow Y$, define $F(\phi)\colon K(Y) \longrightarrow K(X)$ by composition. If f is a rational function on Y, then $\phi \circ f$ is a rational function on X.

We have to check that F is essentially surjective and fully faithful. Suppose that L is a finitely generated field extension of K. Then $L = K(\alpha_1, \alpha_2, \dots, \alpha_n)$. Let $A = K[\alpha_1, \alpha_2, \dots, \alpha_n]$. Let X be any affine variety with coordinate ring A. Then X is irreducible as A is an integral domain and the function field of X is precisely L as this is the field of fractions of A.

The fact that F is fully faithful is proved in the same way as before. □

Proposition 14.12. *Let X and Y be two irreducible varieties.*

Then the following are equivalent

1. *X and Y are birational.*
2. *X and Y contain isomorphic open subsets.*
3. *The function fields of X and Y are isomorphic.*

Proof. We have already seen that (1) and (3) are equivalent and clearly (2) implies (1) (or indeed (3)). It remains to prove that if X and Y are birational then they contain isomorphic open subsets.

Let $\phi\colon X \dashrightarrow Y$ be a birational map with inverse $\psi\colon Y \dashrightarrow X$. Suppose that ϕ is defined on U and ψ is defined on V. Let $U' = \phi^{-1}(V) \subset U$. Let f be the restriction of ϕ to U'. Then $f\colon U' \longrightarrow f(U') \subset V$. Suppose that ψ is represented by (g, V).

The composition $g \circ f\colon U' \longrightarrow U'$ is the identity morphism, as it is the identity on an open subset. Therefore $f(U') = g^{-1}(U')$ is open and so $g : f(U') \longrightarrow U'$ is the inverse of f. Indeed $f \circ g$ and $g \circ f$ are both morphisms and equal to the identity on dense open subsets, so that they are both the identity morphism. So U' and $f(U')$ are isomorphic open subsets. □

Corollary 14.13. *Let X be an irreducible variety.*

Then the following are equivalent

1. *X is rational.*
2. *X contains an open subset of $\mathbb{P}^n$.*

(3) *The function field of X is a purely transcendental extension of K.*

Proof. Immediate from (14.12). $\square$

Let us consider some examples. I claim that the curve $C = V(y^2 - x^2 - x^3)$ is rational. We have already seen that there is a morphism $\mathbb{A}^1 \longrightarrow C$. We want to show that it is a birational map. One way to proceed is to construct the inverse. In fact the inverse map is $C \dashrightarrow \mathbb{A}^1$ given by $(x, y) \longrightarrow y/x$. Another way to proceed is to prove that the function field is purely transcendental. Now the coordinate ring is

$$K[x, y]/\langle y^2 - x^2 - x^3 \rangle.$$

So the fraction field is $K(x, y)$, where $y^2 = x^2 + x^3$. Consider $t = y/x$. I claim that $K(t) = K(x, y)$. Clearly there is an inclusion one way. Now $t^2 = y^2/x^2 = 1 + x$. So $x = t^2 - 1 \in K(t)$. But $y = tx$, so that we do indeed have equality $K(t) = K(x, y)$. Thus C is rational.

Perhaps a more interesting example is to consider the Segre variety $V \subset \mathbb{P}^3$. Consider projection π from a point p of the Segre variety,

$$\pi \colon V \dashrightarrow \mathbb{P}^2.$$

Clearly the only possible point of indeterminancy is the point p. Since a line, not contained in V, meets the Segre variety in at most two points, it follows that this map is one to one outside p, unless that line is contained in V. On the other hand, if $q \in \mathbb{P}^2$, the line $\langle p, q \rangle$ will meet the Segre variety in at least two points, one of which is p.

Now at the point p, there passes two lines l and m (one line of each ruling). These get mapped to two separate points, say q and r. It follows that p is indeed a point of indeterminancy. To proceed further, it is useful to introduce coordinates. Suppose that $p = [0 : 0 : 0 : 1]$, where $V = V(XW - YZ)$.

Now projection from $p \in \mathbb{P}^3$ defines a rational map

$$\phi \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2,$$

whose exceptional locus is a copy of $\mathbb{P}^2$. Indeed the graph of ϕ lies in $\mathbb{P}^3 \times \mathbb{P}^2$ and as before over the point p, we get a copy of the whole of the image $\mathbb{P}^2$, as can be seen by looking at lines through p. Working on the affine chart $W \neq 0$, V is locally defined as $x = yz$. If $[R : S : T]$ are coordinates on $\mathbb{P}^2$, the equations for the blow up of $\mathbb{P}^3$ are given as

$$xS = yR \qquad xT = zR \qquad yT = zS.$$

The blow up of V at p is given as the strict transform of V in the blow up of $\mathbb{P}^3$. We work in the patch $T \neq 0$. Then $x = rz$ and $y = sz$ so that the we get the equation

$$rz - sz^2 = z(r - sz) = 0.$$

Now $z = 0$ corresponds to the whole exceptional locus so that $r = sz$ defines the strict transform. In this case $z = 0$, means $r = 0$, so that we get a line in the exceptional $\mathbb{P}^2$.

In other words the graph of π is the blow up of p, with an exceptional divisor isomorphic to $\mathbb{P}^1$. The graph of π then blows down the strict transform of the two lines. Note that the image of the exceptional divisor, is precisely the line connecting the two points q and r.

To see that π is birational, we write down the inverse,

$$\psi \colon \mathbb{P}^2 \dashrightarrow V.$$

Given $[R : S : T]$, we send this to $[R : S : T : ST/R]$. Clearly this lies on the quadric $XW - YZ$ and is indeed the inverse map. Note that the inverse map blows up q and r then blows down the line connecting them to p.

In fact it turns out that the picture above for rational maps on surfaces is the complete picture.

Theorem 14.14 (Elimination of Indeterminancy)**.** *Let $\phi \colon S \dashrightarrow Z$ be a rational map from a smooth surface.*

Then there is an iterated sequence of blow ups of points $p \colon T \longrightarrow S$ such that the induced rational map $\psi \colon T \longrightarrow Z$ is a morphism.

Theorem 14.15. *Let $\phi \colon S \dashrightarrow T$ be a birational map of smooth surfaces.*

Then there is an iterated sequence of blow ups of points $p \colon W \longrightarrow S$ such that the induced map $q \colon W \longrightarrow T$ is also an iterated sequence of blow up of points.

In fact it turns out that both of these results generalise to all dimensions. In the first result, one must allow blowing up the ideal of any smooth subvariety. In the second result, one must allow mixing up the sequence of blowing up and down, although it is conjectured that the one can perform first a sequence of blow ups and then a sequence of blow downs.

Another way to proceed, is to compute the field of fractions. The coordinate ring on the affine piece $W \neq 0$ is

$$K[x, y, z]/\langle x - yz \rangle = k[y, z].$$

The field of fractions is visibly then $K(y, z)$. However perhaps the easiest way to proceed is to observe that $\mathbb{P}^1 \times \mathbb{P}^1$ contains $\mathbb{A}^1 \times \mathbb{A}^1 \simeq \mathbb{A}^2$, so that the Segre Variety is clearly rational.

In fact it turns out in general to be a vary hard problem to determine which varieties are rational. As an example of this consider Lüroth's problem.

Definition 14.16. *We say that a variety X is* ***unirational*** *if there is a dominant rational map $\phi\colon \mathbb{P}^n \dashrightarrow X$.*

Question 14.17 (Lüroth)**.** *Is every unirational variety rational?*

Note that one way to restate Lüroth's problem is to ask if every subfield of a purely transcendental field extension is purely transcendental. It turns out that the answer is yes in dimension one, in all characteristics. This is typically a homework problem in a course on Galois Theory.

In dimension two the problem is already considerably harder, and it is false if one allows inseparable field extensions. The first step is in fact to establish (14.14) and (14.15).

In dimension three it was shown to be false even in characteristic zero, in 1972, using three different methods.

One proof is due to Artin and Mumford. It had been observed by Serre that the cohomology ring of a smooth unirational threefold is indistinguishable from that of a rational variety (for $\mathbb{P}^3$ one gets $\frac{\mathbb{Z}[x]}{\langle x^3 \rangle}$, and the cohomology ring varies in a very predictable under blowing up and down) except possibly that there might be torsion in $H^3(X, \mathbb{Z})$. They then give an reasonably elementary construction of a threefold with non-zero torsion in H^3.

Another proof is due to Clemens and Griffiths. It is not hard to prove that every smooth cubic hypersurface in $\mathbb{P}^4$ is unirational. On the other hand they prove that some smooth cubics are not rational. To prove this consider the family of lines on the cubic. It turns out that this is a two dimensional family, and that a lot of the geometry of the cubic is controlled by the geometry of this surface.

The third proof is due to Iskovskikh and Manin. They prove that every smooth quartic in $\mathbb{P}^4$ is not rational. On the other hand, some quartics are unirational. In fact they show, in an amazing tour de force, that the birational automorphism group of a smooth quartic is finite. Clearly this means that a smooth quartic is never rational.