13. Projective varieties and schemes

Definition 13.1. *Let R be a ring. We say that R is **graded**, if there is a direct sum decomposition,*

$$R = \bigoplus_{d \in \mathbb{N}} R_d,$$

where each R_d is an additive subgroup of R, such that

$$R_d R_e \subset R_{d+e}.$$

*The elements of R_d are called the **homogeneous elements of order** d.*

*Let R be a graded ring. We say that an R-module M is **graded** if there is a direct sum decomposition*

$$M = \bigoplus_{n \in \mathbb{N}} M_n,$$

compatible with the grading on R in the obvious way,

$$R_d M_n \subset M_{d+n}.$$

*A **morphism of graded modules** is an R-module map $\phi\colon M \longrightarrow N$ of graded modules, which respects the grading,*

$$\phi(M_n) \subset N_n.$$

*A **graded submodule** is a submodule for which the inclusion map is a graded morphism. A **graded ideal** I of R is an ideal, which when considered as a submodule, is a graded submodule.*

Note that the kernel, image and cokernel of a morphism of graded modules is a graded module. Note also that an ideal is a graded ideal iff it is generated by homogeneous elements. Here is the key example:

Example 13.2. *Let R be the polynomial ring over a ring S. Define a direct sum decomposition of R by taking R_n to be the set of homogeneous polynomials of degree n. Given a graded ideal I in R, that is an ideal generated by homogeneous elements of R, the quotient is a graded ring.*

Remark 13.3. *Suppose that R is a graded ring, and that S is a multiplicative set, generated by homogeneous elements. Then R_S is a graded ring, where the grading is given by*

$$\deg f/g = \deg f - \deg g,$$

where, of course, the grading is now given by the integers.

Definition 13.4. *Let $X \subset \mathbb{P}^n$ be a projective variety. The ideal $I(X)$ of X is the ideal generated by the homogeneous polynomials which vanish on X.*

The ***homogeneous coordinate ring*** *of X is equal to the quotient $K[X_0, X_1, \dots, X_n]/I(X)$.*

Note that the homogeneous coordinate ring of X is a graded ring, since the ideal $I(X)$ is homogeneous. Note the following:

Lemma 13.5. *Let I be a homogeneous ideal in a graded ring R.*

Then the radical of I is also a homogeneous ideal.

Proof. Pick $r \in R$, such that $r^n \in I$. Suppose that

$$r = r_0 + r_1 + \cdots + r_k,$$

is the decomposition of r into its homogeneous pieces. We want to prove that r_i belongs to the radical of I. By induction, it suffices to prove that r_k is in the radical. But if we expand r^n, then r_k^n is the only part of degree nk. Since I is homogeneous, it follows that $r_k^n \in I$. Thus r_k is in the radical of I. □

Theorem 13.6. *Let I be a homogeneous ideal in the polynomial ring and let $X = V(I)$.*

If X is not empty then $I(X)$ is equal to the radical of I.

Indeed the proof is the same as before, using (13.5). Note that this establishes a correspondence between projective varieties and homogeneous ideals, with the only twist being that the ideal

$$\langle X_0, X_1, \dots, X_n \rangle,$$

does not correspond to any projective variety. One subtle point is that the homogeneous coordinate ring remembers the embedding, unlike the coordinate ring of an affine variety. Thus there is no correspondence between projective varieties and finitely generated graded K-algebras without nilpotents.

Note that for a projective variety, unlike for an affine variety there are three different ways in which a collection of homogeneous polynomials can cut out X.

Definition 13.7. *Let X be a projective variety and let $F_1, F_2, \dots, F_k$ be a collection of homogenous polynomials. We say that $F_1, F_2, \dots, F_k$ cuts out X*

1. ***set-theoretically*** *if $V(F_1, F_2, \dots, F_k) = X$.*
2. ***scheme-theoretically*** *if for every i, if $f_1, f_2, \dots, f_k$ denotes the dehomogenisation of $F_1, F_2, \dots, F_k$ in the affine piece $X_i \neq 0$, then $\langle f_1, f_2, \dots, f_k \rangle = I(X \cap U_i)$.*

(3) ***ideal-theoretically*** *if* $\langle F_1, F_2, \dots, F_k\rangle = I(X)$.

Note that $X \cap U_i$ is an affine variety in $\mathbb{A}^n$. Of course we are familiar with the first and last notion. A moments thought will convince the reader that the middle notion is intermediary between the other two. Let us see that this notion is really distinct from the other two.

In fact if $F_1, F_2, \dots, F_k$ generate the ideal of X, note that the products $G_{ij} = X_i F_j$, $0 \leq i \leq n$ and $0 \leq j \leq n$ certainly cut out X set-theoretically (essentially because the common vanishing locus of the X_i is empty). Now the products G_{ij} certainly don't generate the ideal of X (indeed, supposing that the degrees of F_j are increasing, it is clear that F_1 is not a combination of the G_{ij} by reasons of degree). On the other hand, on the affine open piece U_i, the dehomogenisation of G_{ij} is the same as the dehomogenisation of F_j. Thus G_{ij} certainly cut out X scheme-theoretically.

It is interesting to go back to some of the morphisms given before and give intrinsic definitions of these maps, at least in characteristic zero.

Let V be a vector space of dimension two. Then there is a natural map

$$V \longrightarrow \mathrm{Sym}^d(V),$$

obtained by sending a vector v to its dth symmetric power, v^d. This induces a map

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^d,$$

obtained by sending the line $[v]$ to the line $[v^d]$.

Lemma 13.8. *Suppose that the characteristic is zero (or more generally coprime to d).*

Then the map defined above is precisely the d-uple embedding.

Proof. Pick a basis e and f of V. Then a general vector in V is of the form $v = ae + bf$. Then we expand $(ae + bf)^d$ using the binomial Theorem.

$$(ae + df)^d = a^d e^d + \binom{d}{1} a^{d-1} b e^{d-1} + \dots + b^d f^d.$$

Thus the given map is

$$[a, b] \longrightarrow [a^d : \binom{d}{1} a^{d-1} b : \binom{d}{2} a^{d-2} b^2 : \dots : b^d].$$

Replacing a by S and b by T, note that the entries gives us a basis for the polynomials of degree d. Thus changing coordinates we get the d-uple embedding. $\square$

Note that this interpretation sheds new light on the fact that the rational normal curve is a determinental variety. Indeed we have identified the rational normal curve as being the locus of rank one symmetric tensors in $\mathrm{Sym}^d(V)$.

Similarly, in characteristic zero, the general d-uple embedding, has the same description. In particular the Veronese surface in $\mathbb{P}^5$, may be identified as the locus of rank one symmetric tensors inside $\mathbb{P}(\mathrm{Sym}^2(V))$, where V is a three dimensional vector space.

Proposition 13.9. *Let $X \subset \mathbb{P}^n$ be a projective variety.*

Then we can embed X into projective space so that X is cut out scheme-theoretically by quadratic equations.

Proof. Suppose that $F_1, F_2, \dots, F_k$ generate the ideal of X. Multiplying each F_i by all monomials of a given degree, we may assume that the degree of each F_i is the same (as above, the new polynomials still cut out X scheme-theoretically). Now consider the d-uple embedding of $\mathbb{P}^n$ into $\mathbb{P}^N$. Let Y be the image of X. Then the polynomials $F_1, F_2, \dots, F_k$ correspond to linear polynomials in $\mathbb{P}^N$. Since the image of $\mathbb{P}^N$ is cut out ideal theoretically by quadrics and the restriction of a quadric to a hyperplane is a quadric, it follows that Y is cut out by quadrics in some linear space contained in $\mathbb{P}^N$. $\square$

Now we turn to the definition of projective schemes. The definition mirrors that for affine schemes. First we start with a graded ring S,

$$S = \bigoplus_{d \in \mathbb{N}} S_d.$$

We set

$$S_+ = \bigoplus_{d > 0} S_d,$$

and we let $\mathrm{Proj}\, S$ denote the set of all homogeneous prime ideals of S, which do not contain S_+. We put a topology on $\mathrm{Proj}\, S$ analogously to the way we put a topology on $\mathrm{Spec}\, S$; if $\mathfrak{a}$ is a homogeneous ideal of S, then we set

$$V(\mathfrak{a}) = \{\, \mathfrak{p} \in \mathrm{Proj}\, S \mid \mathfrak{a} \subset \mathfrak{p} \,\}.$$

The Zariski topology is the topology where these are the closed sets. If $\mathfrak{p}$ is a homogeneous prime ideal, then $S_{(\mathfrak{p})}$ denotes the elements of degree zero in the localisation of S at the set of homogenous elements which do not belong to $\mathfrak{p}$. We define a sheaf of rings $\mathcal{O}_X$ on $X = \mathrm{Proj}\, S$ by considering, for an open set $U \subset X$, all functions

$$s \colon U \longrightarrow \coprod_{\mathfrak{p} \in U} S_{(\mathfrak{p})},$$

such that $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$, which are locally represented by quotients. That is given any point $\mathfrak{q} \in U$, there is an open neighbourhood V of $\mathfrak{p}$ in U and homogeneous elements a and f in S of the same degree, such that for every $\mathfrak{p} \in V$, $f \notin \mathfrak{p}$ and $s(\mathfrak{p})$ is represented by the class of $a/f \in S_{(\mathfrak{p})}$.

Proposition 13.10. *Let S be a graded ring and set $X = \operatorname{Proj} S$.*

(1) *For every $\mathfrak{p} \in X$, the stalk $\mathcal{O}_{X,\mathfrak{p}}$ is isomorphic to $S_{(\mathfrak{p})}$.*
(2) *For any homogeneous element $f \in S_+$, set*

$$U_f = \{\, \mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p} \,\}.$$

Then U_f is open in $\operatorname{Proj} S$, these sets cover X and we have an isomorphism of locally ringed spaces

$$(U_f, \mathcal{O}_X|_{U_f}) \simeq \operatorname{Spec} S_{(f)}.$$

where $S_{(f)}$ consists of all elements of degree zero in the localisation S_f.

In particular $\operatorname{Proj} S$ is a scheme.

Proof. The proof of (1) follows similar lines to the affine case and is left as an exercise for the reader. $U_f = X - V(\langle f \rangle)$ and so U_f is certainly open and these sets certainly cover X. We are going to define an isomorphism

$$(g, g^{\#})\colon (U_f, \mathcal{O}_X|_{U_f}) \longrightarrow \operatorname{Spec} S_{(f)}.$$

If $\mathfrak{a}$ is any homogeneous ideal of S, consider the ideal $\mathfrak{a}S_f \cap S_{(f)}$. In particular if $\mathfrak{p}$ is a prime ideal of S, then $\phi(\mathfrak{p}) = \mathfrak{p}S_f \cap S_{(f)}$ is a prime ideal of $S_{(f)}$. It is easy to see that ϕ is a bijection. Now $\mathfrak{a} \subset \mathfrak{p}$ iff

$$\mathfrak{a}S_f \cap S_{(f)} \subset \mathfrak{p}S_f \cap S_{(f)} = \phi(\mathfrak{p}),$$

so that ϕ is a homeomorphism. If $\mathfrak{p} \in U_f$ then $S_{(\mathfrak{p})}$ and $(S_{(f)})_{\phi(\mathfrak{p})}$ are naturally isomorphic. As in the proof in the affine case, this induces a morphism $g^{\#}$ of sheaves which is easily seen to be an isomorphism. $\square$

Definition 13.11. *Let R be a ring.* ***Projective n-space over R****, denoted $\mathbb{P}^n_R$, is the proj of the polynomial ring $R[x_1, x_2, \dots, x_n]$.*

Note that $\mathbb{P}^n_R$ is a scheme over $S = \operatorname{Spec} R$. Note that we can also define projective n-space $\mathbb{P}^n_S$ over any scheme S. Just pull back $\mathbb{P}^n_{\mathbb{Z}}$ along the unique morphism $S \longrightarrow \operatorname{Spec} \mathbb{Z}$.

We end this section with a result that we could have proved earlier. We show that the category of varieties embeds in a natural way into the category of schemes. We start with the problem of adding the extra points, which are not closed:

Definition-Lemma 13.12. *If X is a topological space, then let $t(X)$ be the set of irreducible closed subsets of X. Then $t(X)$ is naturally a topological space and if we define a map $\alpha\colon X \longrightarrow t(X)$ by sending a point to its closure then α induces a bijection between the closed sets of X and $t(X)$.*

Proof. Observe that

- If $Y \subset X$ is a closed subset, then $t(Y) \subset t(X)$,
- if Y_1 and Y_2 are two closed subsets, then $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$, and
- if Y_α is any collection of closed subsets, then $t(\cap Y_\alpha) = \cap t(Y_\alpha)$.

The defines a topology on $t(X)$ and the rest is clear. □

Theorem 13.13. *Let k be an algebraically closed field. Then there is a fully faithful functor t from the category of varieties over k to the category of schemes over* $\operatorname{Spec} k$*. For any variety V, the set of points of V may be recovered from the closed points of $t(V)$ and the sheaf of regular functions is the restriction of the structure sheaf to the set of closed points.*

Proof. We will show that $(t(V), \alpha_* \mathcal{O}_V)$ is a scheme, where $\mathcal{O}_V$ is the sheaf of regular functions on V. As this problem is local and any variety has an open affine cover, it suffices to prove this when V is an affine variety, with coordinate ring A. Let X be the spectrum of A. We are going to a define a morphism of locally ringed spaces,

$$\beta = (f, f^{\#})\colon (V, \mathcal{O}_V) \longrightarrow (X, \mathcal{O}_X).$$

If $p \in V$, then let $f(p) = m_p \in X$ be the maximal ideal of elements of A vanishing at p. By the Nullstellensatz, f induces a bijection between the closed points of X and the points of V. It is easy to see that f is a homeomorphism onto its image. Now let $U \subset X$ be an open set. We need to define a ring homomorphism

$$f^{\#}(U)\colon \mathcal{O}_X(U) \longrightarrow f_* \mathcal{O}_V(U) = \mathcal{O}_V(f^{-1}(U)).$$

Let $s \in \mathcal{O}_X(U)$. We want to define $r = f^{\#}(U)(s)$. Pick $p \in U$. Then we define $r(p)$ to be the image of $s(m_p) \in A_{m_p}$ inside the quotient

$$A_{m_p}/m_p \simeq k.$$

It is easy to see that r is a regular function and that $f^{\#}(U)$ is a ring isomorphism. This defines $f^{\#}$ and β.

As the irreducible subsets of V are in bijection with the prime ideals of A, it follows that $(X, \mathcal{O}_X)$ is isomorphic to $(t(V), \alpha_* \mathcal{O}_V)$, and so the latter is an affine scheme.

Note that there is a natural inclusion

$$k \subset \mathcal{O}_V(V) = \mathcal{O}_X(X),$$

which associates to a scalar the constant function on V. As $\operatorname{Spec} k$ is affine there is a morphism $X \longrightarrow \operatorname{Spec} k$ and so X is a scheme over $\operatorname{Spec} k$.

Suppose that V and W are quasi-projective varieties. To give a morphism from V to W is the same as to cover V and W by affine varieties V_i and W_i and to give morphisms on each piece which agree on overlaps. Now a morphism $V_i \longrightarrow W_i$ is the same as a k-algebra homomorphism $B_i \longrightarrow A_i$ between the two coordinate rings, which induces a morphism between the two corresponding varieties $X_i = t(V_i)$ and $Y_i = t(W_i)$. It is easy to see that these glue to give a morphism $X = t(V) \longrightarrow Y = t(Y)$ This defines the functor t and it is easy to check that t is fully faithful. $\square$