12. Fibre products of schemes

We start with some basic properties of schemes.

Definition 12.1. *We say that a scheme is* ***connected*** *(respectively* ***irreducible****) if its topological space is connected (respectively irreducible).*

Definition 12.2. *We say that a scheme is* ***reduced*** *if $\mathcal{O}_X(U)$ contains no nilpotent elements, for every open set U.*

Remark 12.3. *It is straightforward to prove that a scheme is reduced iff the stalk of the structure sheaf at every point contains no nilpotent elements.*

Definition 12.4. *We say that a scheme X is* ***integral*** *if for every open set $U \subset X$, $\mathcal{O}_X(U)$ is an integral domain.*

Proposition 12.5. *A scheme X is integral iff it is irreducible and reduced.*

Proof. Suppose that X is integral. Then X is surely reduced. Suppose that X is reducible. Then we can find two non-empty disjoint open sets U and V. But then

$$\mathcal{O}_X(U \cup V) \simeq \mathcal{O}_X(U) \times \mathcal{O}_X(V),$$

which is surely not an integral domain.

Now suppose that X is reduced and irreducible. Let $U \subset X$ be an open set and suppose that we have f and $g \in \mathcal{O}_X(U)$ such that $fg = 0$. Set

$$Y = \{\, x \in U \mid f_x \in m_x \,\} \qquad \text{and} \qquad Z = \{\, x \in U \mid g_x \in m_x \,\}.$$

Then Y and Z are both closed and by assumption $Y \cup Z = U$. As X is irreducible, one of Y and Z is the whole of U, say Y. We may assume that $U = \operatorname{Spec} A$ is affine. But then $f \in A$ belongs to the intersection of all the prime ideals of A, which is the zero ideal, as A contains no nilpotent elements. □

Definition 12.6. *We say that a scheme X is* ***locally Noetherian****, if there is an open affine cover, such that the corresponding rings are Noetherian. If in addition the topological space is compact, then we say that X is* ***Noetherian****.*

Remark 12.7. *There are examples of schemes whose topological space is Noetherian which are not Noetherian schemes.*

A key issue in this definition is whether or not we can replace an open cover, by every affine cover.

Proposition 12.8. *A scheme X is locally Noetherian iff for every open affine $U = \operatorname{Spec} A$, A is a Noetherian ring.*

Proof. It suffices to prove that if X is locally Noetherian, and $U = \operatorname{Spec} A$ is an open affine subset then A is a Noetherian ring.

We first show that U is locally Noetherian. Suppose that $V = \operatorname{Spec} B$ is an open affine on X where B is a Noetherian ring. Then $U \cap V$ can be covered by open sets of the form $V_f = spB_f$, where $f \in B$. As B is a Noetherian ring then so is B_f. As open sets of the form V cover X, U is covered by open affines, which are the spectra of Noetherian rings. So U is locally Noetherian.

Replacing X by U, we are reduced to proving that if $X = \operatorname{Spec} A$ locally Noetherian then A is Noetherian. Let $V = \operatorname{Spec} B$, be an open subset of X, where B is a Noetherian ring. Then there is an element $f \in A$ such that $U_f \subset V$. Let g be the image of f in B. As

$$X \supset U_f = U_g \subset V,$$

we have an isomorphism of rings $A_f \simeq B_g$, whence A_f is Noetherian. So we can cover X by open subsets $U_f = \operatorname{Spec} A_f$, with A_f Noetherian. As X is compact, we may assume that we have a finite cover. Now apply (12.9). $\square$

Lemma 12.9. *Let A be a ring, and let $f_1, f_2, \dots, f_r$ be elements of A which generate the unit ideal.*

If A_{f_i} is Noetherian, for $1 \leq i \leq r$ then so is A.

Proof. Suppose that we have an ascending chain of ideals,

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \mathfrak{a}_3 \subset \dots,$$

of A. Then for each i,

$$\phi_i(\mathfrak{a}_1) \cdot A_{f_i} \subset \phi_i(\mathfrak{a}_2) \cdot A_{f_i} \subset \phi_i(\mathfrak{a}_3) \cdot A_{f_i} \subset \dots,$$

is an ascending chain of ideals inside A_{f_i}, where $\phi_i \colon A \longrightarrow A_{f_i}$ is the natural map. As each A_{f_i} is Noetherian, all of these chains stabilise. But then the first chain stabilises, by (12.10). $\square$

Lemma 12.10. *Let A be a ring, and let $f_1, f_2, \dots, f_r$ be elements of A which generate the unit ideal. Suppose that $\mathfrak{a}$ is an ideal and let $\phi_i \colon A \longrightarrow A_{f_i}$ be the natural maps. Then*

$$\mathfrak{a} = \bigcap_{i=1}^{r} \phi_i^{-1}(\phi_i(\mathfrak{a}) \cdot A_{f_i}).$$

Proof. The fact that the LHS is included in the RHS is clear. Conversely suppose that b is an element of the RHS. In this case

$$\phi_i(b) = \frac{a_i}{f^{n_i}},$$

for some $a_i \in \mathfrak{a}$ and some positive integer n_i. As there are only finitely many indices, we may assume that $n = n_i$ is fixed. But then

$$f^{m_i}(f^n b - a) = 0,$$

for $1 \leq i \leq r$. Once again, we may assume that $m = m_i$ is fixed. It follows that $f_i^N b \in \mathfrak{a}$, for $1 \leq i \leq r$, where $N = n + m$. Let I be the ideal generated by the Nth powers of $f_1, f_2, \dots, f_r$. As the radical of I contains 1, I contains 1. Hence we may write

$$1 = \sum_i c_i f_i^N.$$

But then

$$b = \sum_i c_i f_i^N b \in \mathfrak{a}. \qquad \square$$

Definition 12.11. *A morphism $f \colon X \longrightarrow Y$ is **locally of finite type** if there is an open affine cover $V_i = \operatorname{Spec} B_i$ of Y, such that $f^{-1}(V_i)$ is a union of affine sets $U_{ij} = \operatorname{Spec} A_{ij}$, where each A_{ij} is a finitely generated B_i-algebra. If in addition, we can take U_{ij} to be a finite cover of $f^{-1}(V_i)$, then we say that f is of **finite type**.*

Definition 12.12. *We say that a morphism $f \colon X \longrightarrow Y$ is **finite** if we may cover Y by open affines $V_i = \operatorname{Spec} B_i$, such that $f^{-1}(V_i) = \operatorname{Spec} A_i$ is an affine set, where A_i is a finitely generated B_i-module.*

In both cases, it is straightforward to prove that we can take V_i to be any affine subset of Y.

Example 12.13. *Let*

$$f \colon \mathbb{A}^1_k - \{0\} \longrightarrow \mathbb{A}^1_k,$$

by the natural map given by the natural localisation map

$$k[x] \longrightarrow k[x]_x.$$

As an algebra over $k[x]$, the ring $k[x]_x \simeq k[x, x^{-1}]$ is generated by x^{-1}, so that f is of finite type. However the $k[x]$-module $k[x, x^{-1}]$ is not finitely generated (there is no way to generate all the negative powers of x), so that f is not finite.

Definition 12.14. *Let X be a scheme and U an open subset of X. Then the pair $(U, \mathcal{O}_U = \mathcal{O}_X|_U)$ is a scheme, which is called an **open subscheme** of X. An **open immersion** is a morphism $f\colon X \longrightarrow Y$ which induces an isomorphism of X with an open subset of Y.*

Definition 12.15. *A **closed immersion** is a morphism of schemes $\phi = (f, f^{\#})\colon Y \longrightarrow X$ such that f induces a homeomorphism of Y with a closed subset of X and futhermore the map $f^{\#}\colon \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is surjective. A **closed subscheme** of a scheme X is an equivalence class of closed immersions, where we say that two closed immersions $f\colon Y \longrightarrow X$ and $f'\colon Y' \longrightarrow X$ are equivalent if there is an isomorphism $i\colon Y' \longrightarrow Y$ such that $f' = f \circ i$.*

Despite the seemingly tricky nature of the definition of a closed immersion, in fact it is easy to give examples of closed subschemes of an affine variety.

Lemma 12.16. *Let A be a ring and let $\mathfrak{a}$ be an ideal of A. Let $X =$ $\operatorname{Spec} A$ and $Y = \operatorname{Spec} A/\mathfrak{a}$.*

Then Y is a closed subscheme of X.

Proof. The quotient map map $A \longrightarrow A/\mathfrak{a}$ certainly induces a morphism of schemes $\phi\colon Y \longrightarrow X$. Indeed f is certainly a homeomorphism of Y with $V(\mathfrak{a})$ and $f^{\#}\colon \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is surjective as the map on stalks is induced by the quotient map, which is surjective. □

In fact, it turns out that every closed subscheme of an affine scheme is of this form. It is interesting to look at some examples.

Example 12.17. *Let $X = \mathbb{A}^2_k$. First consider $\mathfrak{a} = \langle y^2 \rangle$. The support of Y is the x-axis. However the scheme Y is not reduced, even though it is irreducible. It is clear from this example that in general there are many closed subschemes with the same support (equivalently there are many ideals with the same radical). Now consider the ideal $\langle x^2, xy, y^2 \rangle$, the double of the maximal ideal of a point. Similarly consider $\langle x, y^2 \rangle$. Finally consider $\langle x^2, xy \rangle$. The support of this ideal is the y-axis. But this time the only local ring which has nilpotents is the local ring of the origin. We call the origin an **embedded point.***

Definition 12.18. *Let V be an irreducible affine variety with coordinate ring A and let W be a closed irreducible subvariety, defined by the prime ideal $\mathfrak{p}$. Then we can associate two affines schemes $Y \subset X$ to $W \subset V$. Let $X = \operatorname{Spec} A$ and define Y by $\mathfrak{p}$. The n**th infinitessimal neighbourhood of** Y **in** X, denoted Y_n, is the closed subscheme of X corresponding to $\mathfrak{p}^n$.*

Note that the nth infinitessimal neighbourhood of Y in X is a closed subscheme whose support coincides with Y, but whose structure sheaf contains lots of nilpotent elements. As the name might suggest, Y_n carries more information about how Y sits inside X, than does Y itself.

Note that if a scheme X has a topological space with one point, then X must be affine, and the stalk of the structure sheaf at the unique point completely determines X, and this ring has exactly one prime ideal. Moreover a morphism of X into another scheme Y, is equivalent to picking a point y of Y and a morphism of local rings

$$\mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}.$$

But to give a morphism of local rings is the same as to give an inclusion of the quotients of the maximal ideals. Thus to give a morphism of $X = \{x\}$ into Y, sending x to y, we need to specify an inclusion of the residue field of x into the residue field of y.

The main result of this section is:

Theorem 12.19. *The category of schemes admits fibre products.*

A key part of the proof is to pass from the local case (in which case all three schemes are affine) to the global case. To do this, we need to be able to construct morphisms, by constructing them locally. We will need:

Theorem 12.20. *Let $f_i\colon U_i \longrightarrow Y$ be a collection of morphisms of schemes, with a varying domain, but a fixed target.*

Suppose that for each pair of indices i and j, we are given open subsets $U_{ij} \subset U_i$, and isomorphisms $\phi_{ij}\colon U_{ij} \longrightarrow U_{ji}$, such that $f_i|_{U_{ij}} = f_j \circ \phi_{ij}\colon U_{ij} \longrightarrow Y$ and

$$\phi_{ik} = \phi_{jk} \circ \phi_{ij},$$

on the intersection $U_{ij} \cap U_{ik}$, for all i, j and k (we adopt the convention that $U_{ii} = U_i$, so that ϕ_{ii} is the identity and moreover $\phi_{ij}^{-1} = \phi_{ji}$).

Then there is a morphism of schemes $f\colon X \longrightarrow Y$, open immersions $\psi_i\colon U_i \longrightarrow X$, whose images cover X, such that $f_i = f \circ \psi_i\colon U_i \longrightarrow Y$ and $\psi_i|_{U_{ij}} = \psi_j \circ \phi_{ij}\colon U_{ij} \longrightarrow Y$.

X is unique, up to unique isomorphism, with these properties.

We prove (12.20) in two steps (one of which can be further broken down into two substeps):

- Construct the scheme X.
- Construct the morphism f.

In fact, having constructed X, it is straightforward to construct f. Since a scheme consists of two parts, a topological space and a sheaf, we can break the first step into two smaller pieces:

- Construct the underlying topological space.
- Construct the structure sheaf.

We first show how to patch a sheaf, which is the hardest part:

Lemma 12.21. *Let X be a topological space, and let $\{X_i\}$ be an open cover of X. Suppose that we are given sheaves $\mathcal{F}_i$ on X_i and for each i and j an isomorphism*

$$\phi_{i,j} \colon \mathcal{F}_i|_{X_{ij}} \longrightarrow \mathcal{F}_j|_{X_{ij}},$$

such that

$$\phi_{ik} = \phi_{jk} \circ \phi_{ij},$$

on the triple intersection X_{ijk}, for all i, j and k.

Then there is a sheaf $\mathcal{F}$ on X, together with isomorphisms, $\psi_i \colon \mathcal{F}|_{X_i} \longrightarrow \mathcal{F}_i$, which satisfy $\psi_j = \phi_{ij} \circ \psi_i$. Further $\mathcal{F}$ is unique up to unique isomorphism, with these properties.

Proof. We just show how to define $\mathcal{F}$ and leave the rest to the interested reader. Let $U \subset X$ be any open set, and let $U_i = U \cap X_i$.

$$\mathcal{F}(U) = \{\, (s_i) \in \prod_i \mathcal{F}(U_i) \mid \phi_{ij}(s_i|_{U_{ij}}) = s_j|_{U_{ji}} \,\}. \qquad \square$$

Using (12.21), one can put a natural scheme structure on any closed subset of a scheme (natural means the smallest possible scheme structure):

Definition-Lemma 12.22. *Let X be scheme and let Y be a closed subset. Then Y has a unique reduced subscheme structure, called the **reduced induced subscheme structure**.*

Proof. We first assume that $X = \operatorname{Spec} A$ is affine. Let $\mathfrak{a}$ be the ideal obtained by intersecting all the prime ideals in Y. Then $\mathfrak{a}$ is the largest ideal for which $V(\mathfrak{a}) = Y$. The induced scheme structure on Y is reduced, that is the stalks of $\mathcal{O}_Y$ have no nilpotent elements, as $\mathfrak{a}$ is a radical ideal.

Now suppose that X is an arbitrary scheme. For each open affine subset $U_i \subset X$, let $Y_i \subset U_i$ be the reduced induced subscheme structure on $Y \cap U_i$. This gives us a sheaf $\mathcal{O}_{Y_i}$ on each Y_i and we want to construct a sheaf $\mathcal{O}_Y$ on the whole of Y. By (12.21) it suffices to prove that the sheaves $\mathcal{O}_{Y_i}$ agree on overlaps.

It is not hard to reduce to the case where $U = \operatorname{Spec} A$, $V = \operatorname{Spec} A_f$. We want to show that the reduced induced subscheme structure on V is the same as resricting the reduced induced subscheme structure from U to V. But this is the same as to say that if $\mathfrak{a}$ is the intersection of those prime ideals of A which are contained in Y, then $\mathfrak{a}A_f$ is the intersection of those prime ideals of A_f which are contained in Y, which is clear. $\square$

The next step is to bump this up to schemes:

Lemma 12.23. *Suppose that we are given schemes U_i, and subschemes $U_{ij} \subset U_i$, together with isomorphisms,*

$$\phi_{ij} \colon U_{ij} \longrightarrow U_{ji},$$

which satisfy

$$\phi_{ik} = \phi_{jk} \circ \phi_{ij},$$

on the intersection $U_{ij} \cap U_{ik}$, for all i, j and k.

Then there is a scheme X and open immersions $\psi_i \colon U_i \longrightarrow X$, whose images cover X, which satsify $\psi_i|_{U_{ij}} = \phi_{ij} \circ \psi_i \colon U_{ij} \longrightarrow X$.

Proof. We first construct the topological space X. Let

$$X = \coprod_i U_i / \sim \qquad \text{where} \qquad x_i \in U_{ij} \sim x_j \in U_{ji} \text{ iff } \phi_{ij}(x_i) = x_j.$$

Here $\sim$ denotes the equivalence relation generated by the rule on the RHS, and X is just the quotient topological space (which always exists). Note that

$$X_i = U_i / \sim,$$

is an open subset of X and there are homeomorphisms $\phi_i \colon U_i \longrightarrow X_i$. Now construct a sheaf $\mathcal{O}_X$ on X, using (12.21). This gives us a locally ringed space $(X, \mathcal{O}_X)$ and the remaining properties can be easily checked. $\square$

There are a couple of interesting examples of the construction of schemes. The first is to take U_{ij} empty (so that there are no patching conditions at all). The resulting scheme is called the disjoint union and is denoted

$$\coprod_i X_i.$$

Another more interesting example proceeds as follows. Take two copies U_1 and U_2 of the affine line. Let $U_{12} = U_{21}$ be the complement of the origin, and let ϕ_{12} be the identity. Then X is obtained by identifying every point, except the origin. Note that this is like the classical construction of a topological space, which is locally a manifold, but which is not Hausdorff. Of course no scheme is ever Hausdorff (apart from the most trivial examples) and it turns out that there is an appropriate condition for schemes (and in fact morphisms of schemes) which corresponds to the Hausdorff condition for topological spaces.

Finally we turn to the problem of glueing morphisms, which is the easiest bit:

Proof of (12.20). Let X be the scheme constructed in (12.23). It is clear that to give a morphism $f\colon X \longrightarrow Y$ is the same as to give morphisms $f_i\colon X_i \longrightarrow Y$, compatible on overlaps. $\square$

Lemma 12.24. *Let X and Y be schemes over S. Suppose that X has an open cover $\{X_i\}$ such that the fibre product $F_i = X_i \underset{S}{\times} Y$ exists.*

Then the fibre product $F = X \underset{S}{\times} Y$ exists.

Proof. Let $p_i\colon F_i \longrightarrow X$ be the natural morphism and let $F_{ij} = p_i^{-1}(F_i \cap F_j)$. Note that F_{ij} is isomorphic to the fibre product of $F_i \cap F_j$ and Y over S. Indeed if Z maps to F_{ij} and Y over S, it maps to X and Y over S. But then Z maps to F_i, by the universal property of the fibre product. It is clear that the image of Z lands in F_{ij}, so that F_{ij} is the fibre product. But then there are natural isomorphisms $\phi_{ij}\colon F_{ij} \longrightarrow F_{ji}$ such that

$$\phi_{ik} = \phi_{jk} \circ \phi_{ij},$$

on the intersection $F_{ij} \cap F_{ik}$, for all i, j and k, and $p_i|_{F_{ij}} = p_j \circ \phi_{ij}$.

(12.20) implies that we may patch F_i to a scheme F, and patch the morphisms $F_i \longrightarrow X$ to a morphism $F \longrightarrow X$. Similarly we may construct a morphism $F \longrightarrow Y$, from the individual morphisms $q_i\colon F_i \longrightarrow Y$.

Now suppose we are given $Z \longrightarrow X$ and $Z \longrightarrow Y$ morphisms over S. The open cover $\{X_i\}$ induces an open cover $\{Z_i\}$ of Z. We get morphisms $Z_i \longrightarrow F_i$, by the universal property of F_i and so we get morphisms $Z_i \longrightarrow F$ by composition. It is easy to check that these patch to a morphism $Z \longrightarrow F$. But then F is the fibre product. $\square$

Proof of (12.19). Let X and Y be two schemes over S. We want to construct the fibre product.

First suppose that $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and $S = \operatorname{Spec} R$. Then there are ring homomorphisms $R \longrightarrow A$ and $R \longrightarrow B$ and so A and B are R-algebras. As $C = A \underset{R}{\otimes} B$ is the pushout in the category of rings, it follows that $Z = \operatorname{Spec} C$ is the fibre product in the category of affine schemes; in fact it is also the fibre product in the category of schemes, since a morphism to an affine variety is the same as a ring homomorphism the other way on global sections.

We now bump this result up to the global case. First suppose that S and Y are affine. Since an arbtirary scheme X can be covered by open affines $\{X_i\}$, (12.24) implies that the fibre product of X and Y over S exists.

Now suppose that S is affine. Since Y can be covered by open affines $\{Y_j\}$ and the fibre product is obviously symmetric in X and Y, (12.24) implies that the fibre product of X and Y over S exists.

Now take an affine cover S_i of S. Let X_i and Y_i be the inverse image of S_i (meaning take the open subscheme on the open set $p_j^{-1}(S_i)$). Then the fibre product $X_i \underset{S}{\times} Y_i$ exists. But in fact this is also a fibre product for $X_i \underset{S}{\times} Y$, since anything lying over X_i automatically lies over Y_i. Since X_i forms an open cover of X we are done by one more application of (12.24). $\square$

It turns out that the fibre product is extremely useful.

Definition 12.25. *Let $f\colon X \longrightarrow S$ be a morphism of schemes, and let $s \in S$ be a point of S. The **fibre over** s is the fibre product over the morphism f and the inclusion of s in S, where the point s is given a scheme structure by taking the residue field $\kappa(s)$.*

It is interesting to see what happens in some specific examples. First consider a family of conics in the plane,

$$X = \operatorname{Spec} \frac{k[x,y,t]}{\langle ty - x^2\rangle}.$$

The inclusion

$$k[t] \longrightarrow \frac{k[x,y,t]}{\langle ty - x^2\rangle},$$

realises X as a family over the affine line over k,

$$f\colon X \longrightarrow \mathbb{A}^1_k.$$

Pick a point $p \in \mathbb{A}^1$. If the point is maximal, this is the same as picking a scalar, and of course the residue field is nothing more than k. If we pick a non-zero scalar a, then we just get the conic defined by $ay - x^2$ in $k[x,y]$ (since tensoring by k won't change anything),

$$X_p = \operatorname{Spec} \frac{k[x,y]}{\langle ay - x^2\rangle}.$$

But now suppose that $a = 0$. In this case the above reduces to

$$X_0 = \operatorname{Spec} \frac{k[x,y]}{\langle x^2\rangle},$$

a double line. It is also interesting to consider the fibre over the generic point ξ, corresponding to the maximal ideal $\langle 0\rangle$. In this case the residue field is $k(t)$, and the **generic fibre** is

$$X_\xi = \operatorname{Spec} \frac{k(t)[x,y]}{\langle ty - x^2\rangle},$$

which is the conic $V(ty - x^2) \subset \mathbb{A}^2_{k(t)}$ over the field $k(t)$.

Similarly, if we pick the family

$$X = \operatorname{Spec} \frac{k[x, y, t]}{\langle xy - t \rangle}.$$

then, for $a \neq 0$, the fibre is a smooth conic, but for $t = 0$ the fibre is a pair of lines.

Once again, the point is that there are some more exotic examples, which can be treated in the same fashion. Consider for example $\operatorname{Spec} \mathbb{Z}[x]$. Once again this is a scheme over $\operatorname{Spec} \mathbb{Z}$, and once again it is interesting to compute the fibres. Suppose first that we take the generic point. Then this has residue field $\mathbb{Q}$. If we tensor $\mathbb{Z}[x]$ by $\mathbb{Q}$, then we get $\mathbb{Q}[x]$. If we take spec of this, we get the affine line over $\mathbb{Q}$. Now suppose that we take a maximal ideal $\langle p \rangle$. In this case the residue field is $\mathbb{F}_p$ the finite field with p elements. Tensoring by this field we get $\mathbb{F}_p[x]$ and taking spec we get the affine line over the finite field with p elements.

It is also possible to figure out all the prime ideals in $\mathbb{Z}[x]$. They are

(1) $\langle 0 \rangle$
(2) $\langle p \rangle$, p a prime number.
(3) $\langle f(x) \rangle$, $f(x)$ irreducible over $\mathbb{Q}$, with content one,
(4) maximal ideals of the form $\langle p, f(x) \rangle$, where $f(x)$ is a monic polynomial whose reduction modulo p is irreducible.

Note that the zero ideal is the generic point, and the closure of the ideal $\langle p \rangle$ is the fibre over the same ideal downstairs. The closure of an ideal of type (3) is perhaps the most interesting. It will consists of all maximal points $\langle p, g \rangle$, where g is a factor of f inside $\overline{\mathbb{F}}_p$.

It is now possible to consider closed subschemes of $\mathbb{A}^1_{\mathbb{Z}}$. For example consider

$$X = \operatorname{Spec} \frac{\mathbb{Z}[x]}{\langle 3x - 16 \rangle}.$$

Fibre by fibre, we get a collection of subschemes of $\mathbb{A}^1_{\mathbb{F}_p}$. If we reduce modulo 5, that is tensor by $\mathbb{F}_5$, then we get

$$X = \operatorname{Spec} \frac{\mathbb{F}_5[x]}{\langle 3x - 1 \rangle},$$

a single point. However something strange happens over the prime 3, since we get an equation which cannot be satifisied. If we think of this as the graph of the rational map $16/3$, then we have a pole at 3, which cannot be removed. Of course over 2, this rational function is zero.

Now suppose that we consider $x^2 - 3$. Then we get a conic. In fact, this is the same as considering

$$\frac{\mathbb{Z}[x]}{\langle x^2 - 3\rangle} = \mathbb{Z}[\sqrt{3}].$$

So the seemingly strange picture we had before becomes a little more clear. Now suppose that we consider a plane conic in $\mathbb{A}^2_{\mathbb{Z}}$,

$$X = \operatorname{Spec} \frac{\mathbb{Z}[x,y]}{\langle x^2 - y^2 - 5\rangle}.$$

Over the typical prime, we get a smooth conic in the corresponding affine plane over a finite field. But now consider what happens over $\langle 2\rangle$ and $\langle 5\rangle$. Modulo two, we have

$$x^2 - y^2 - 5 = (x + y + 1)^2,$$

and modulo 5 we have

$$x^2 - y^2 - 5 = (x - y)(x + y).$$

Thus we get a double line over $\langle 2\rangle$ and a pair of lines over $\langle 5\rangle$.

Let us return to the case of $x^2 - 3$, and consider the residue fields. Recall that there are three cases.

(1) If p divides the discriminant of $K/\mathbb{Q}$ (which in this case is 12), that is $p = 2$ or 3, then the ideal $\langle p\rangle$ is a square in A.

$$\langle 2\rangle A = (\langle 1 + \sqrt{3}\rangle)^2,$$

and

$$\langle 3\rangle A = (\langle \sqrt{3}\rangle)^2.$$

(2) If 3 is a square modulo p, the prime $\langle p\rangle$ factors into a product of distinct primes,

$$\langle 11\rangle A = \langle 4 + 3\sqrt{3}\rangle\langle 4 - 3\sqrt{3}\rangle,$$

or

$$\langle 13\rangle A = \langle 4 + \sqrt{3}\rangle\langle 4 - \sqrt{3}\rangle,$$

(3) If $p > 3$ and 3 is not a square mod p (e.g $p = 5$ and 7), the ideal $\langle p\rangle$ is prime in A.

Let us consider the coordinate rings in all three cases. In the first case we get

$$A/\mathfrak{p}^2,$$

and the residue field is $\mathbb{F}_p$. In the second case there are two points with coordinate rings $\mathbb{F}_p$. Finally in the third case there is a single point with coordinate ring

$$\mathbb{F}_p^2,$$

the unique finite field with p^2 elements. Note that in all three cases, the coordinate ring of the inverse image has length two over the coordinate ring of the base (in our case $\mathbb{F}_p$). In fact this is the general picture. Finite maps have a degree, and the length of the coordinate ring over the base is equal to this degree.

Another useful way to think of the fibre product, is as a base change. In arithmetic, one always wants to compare what happens over different fields, or even different rings.

Definition 12.26. *Let S be a scheme. As $\operatorname{Spec}\mathbb{Z}$ is a terminal object in the category of schemes, there is a unique morphism $S \longrightarrow \operatorname{Spec}\mathbb{Z}$* ***Affine** n**-space*** *over S is the scheme obtained by base change from $\mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[x_1, x_2, \dots, x_n]$, so that*

$$\mathbb{A}^n_S = \mathbb{A}^n_{\mathbb{Z}} \underset{\operatorname{Spec}\mathbb{Z}}{\times} S.$$

Now consider an interesting example over a non-algebraically close field. Consider the inclusion $\mathbb{R} \longrightarrow \mathbb{C}$. This gives a morphism of schemes,

$$f\colon X = \operatorname{Spec}\mathbb{C} \longrightarrow Y = \operatorname{Spec}\mathbb{R},$$

where X and Y are schemes with only one point, but the first has sheaf of rings given $\mathbb{C}$ and the second $\mathbb{R}$. Now consider what happens when we make the base change f over f. Then we get a scheme

$$X \underset{Y}{\times} X.$$

Note that this has degree two over X. Since $\mathbb{C}$ is algebraically closed, in fact this must consist of two points, even though f only has one point in the fibre. Algebraically,

$$\mathbb{C} \underset{\mathbb{R}}{\otimes} \mathbb{C} \simeq \mathbb{C}^2,$$

and the spectrum has two points.

In particular, the property of being irreducible is not preserved by base change. Consider also the example of $\operatorname{Spec} k[x,t]/\langle x^2 - t\rangle \subset \mathbb{A}^2_k$ over the affine line, with coordinate t, say over an algebraically closed field k. Then the fibre over every closed point, except zero, is reducible. But the fibre over the generic point is irreducible, since $x^2 - t$ won't factor, even if you invert every polynomial in t. However suppose that we make a base change of the affine line by the affine line given by

$$\mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k \qquad \text{given by} \qquad t \longrightarrow t^2.$$

After base change, the new scheme is given by $x^2 - t^2$. But this factors, even over the generic point

$$x^2 - t^2 = (x - t)(x + t).$$

Definition 12.27. *Let X be a scheme over a field k. We say that X is* ***geometrically irreducible*** *if $X \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} \bar{k}$ is irreducible.*

Note that the property of being geometrically irreducible is preserved under base change.