11. Schemes

To define schemes, just as with algebraic varieties, the idea is to first define what an affine scheme is, and then realise an arbitrary scheme, as something which is locally an affine scheme. The definition of an affine scheme is motivated by the correspondence between affine varieties and finitely generated algebras over a field, without nilpotents. The idea is that we should be able to associate to any ring R, a topological space X, and a set of continuous functions on X, which is equal to R. In practice this is too much to expect and we need to work with a slightly more general object than a continuous function.

Now if X is an affine variety, the points of X are in correspondence with the maximal ideals of the coordinate ring $A = A(X)$. Unfortunately if we have two arbitrary rings R and S, then the inverse image of a maximal ideal won't be maximal. However it is easy to see that the inverse image of a prime ideal is a prime ideal.

Definition 11.1. *Let R be a ring. $X = \operatorname{Spec} R$ denotes the set of prime ideals of R. X is called the **spectrum** of R.*

Note that given an element of R, we may think of it as a function on X, by considering it value in the quotient.

Example 11.2. *It is interesting to see what these functions look like in specific cases. Suppose that we take $X = \operatorname{Spec} k[x, y]$. Now any element $f = f(x, y) \in k[x, y]$ defines a function on X. Suppose that we consider a maximal ideal of the form $\mathfrak{p} = \langle x - a, x - b\rangle$. Then the value of f at $\mathfrak{p}$ is equal to the class of f inside the quotient*

$$R/\mathfrak{p} = \frac{k[x, y]}{\langle x - a, x - b\rangle}.$$

If we identify the quotient with k, under the obvious identification, then this is the same as evaluating f at (a, b). Now consider $\mathbb{Z}$. Suppose that we choose an element $n \in \mathbb{Z}$. Then the value of n at the prime ideal $\mathfrak{p} = \langle p\rangle$ is equal to the value of n modulo p. For example, consider $n = 60$. Then the value of this function at the point 7 is equal to $60 \mod 7 = 4 \mod 7$. Moroever 60 has zeroes at 2, 3 and 5, where both 3 and 5 are ordinary zeroes, but 2 is a double zero.

Suppose that we take the ring $R = k[x]/\langle x^2\rangle$. Then the spectrum contains only one element, the prime ideal $\langle x\rangle$. Consider the element $x \in R$. Then x is zero on the unique element of the spectrum, but it is not the zero element of the ring.

Now we wish to define a topology on the spectrum of a ring. We want to make the functions above continuous. So given an element

$f \in R$, we want the set

$$\{\, \mathfrak{p} \in \operatorname{Spec} R \,|\, f(\mathfrak{p}) = 0 \,\} = \{\, \mathfrak{p} \in \operatorname{Spec} R \,|\, \langle f \rangle \subset \mathfrak{p} \,\},$$

to be closed. Given that any ideal $\mathfrak{a}$ is the union of all the principal ideals contained in it, so that the set of prime ideals which contain $\mathfrak{a}$ is equal to the intersection of prime ideals which contain every principal ideal contained in $\mathfrak{a}$ and given that the intersection of closed sets is closed, we have an obvious candidate for the closed sets:

Definition 11.3. *The **Zariski topology** on X is given by taking the closed sets to be*

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subset \mathfrak{p}\,\},$$

where $\mathfrak{a}$ is any ideal of R.

Lemma 11.4. *Let R be a ring.*

Then $X = \operatorname{Spec} R$ is a topological space. Moreover the open sets

$$U_f = \{\, \mathfrak{p} \in R \,|\, f \notin \mathfrak{p} \,\},$$

form a base for the topology.

Proof. Easy check. □

By what we said above, the Zariski topology is the weakest topology so that the zero sets of $f \in R$ are closed.

Example 11.5. *Let k be a field. Then $\operatorname{Spec} k$ consists of a single point. Now consider $\operatorname{Spec} k[x]$. If k is an algebraically closed field, then by the Nullstellensatz, the maximal ideals are in correspondence with the points of k. However, since $k[x]$ is an integral domain, the zero ideal is a prime ideal. Since $k[x]$ is a PID, the proper closed sets of X consist of finite unions of maximal ideals. The closure of the point $\xi = \langle 0 \rangle$ is then the whole of X. In particular, not only is the Zariski topology, for schemes, not Hausdorff or T_2, it is not even T_1. Now consider $k[x, y]$, where k is an algebraically closed field. Prime ideals come in three types. The maximal ideals correspond to points of k^2. The zero ideal, whose closure consists of the whole of X. But there are also the prime ideals which correspond to prime elements $f \in k[x, y]$. The zero locus of f is then an irreducible curve C, and in fact the closure of the point $\xi = \langle f \rangle$ is then the curve C. The proper closed sets thus consist of a finite union of maximal ideals, union infinite sets of the maximal ideals which consist of all points belonging to an affine curve C, together with the ideal of each such curve.*

Now suppose that k is not algebraically closed. For example, consider $\operatorname{Spec} \mathbb{R}[x]$. As before the closure of the zero ideal consists of the whole *of X. The maximal ideals come in two flavours. First there are the ideals $\langle x-a\rangle$, where $a\in\mathbb{R}$. But in addition there are also the ideals*

$$\langle x^2+ax+b=(x-\alpha-i\beta)(x-\alpha+i\beta)\rangle,$$

where a, b, α and $\beta>0$ are real numbers, so that $b^2-4a<0$.

There is a very similar (but more complicated) picture inside $\operatorname{Spec}\mathbb{R}[x,y]$. *The set $V(x^2+y^2=-1)$ does not contain any ideals of the first kind, but it contains many ideals of the second kind. In the classical picture, the conic does $x^2+y^2=-1$ does not contain any points but it does contain many points if you include all prime ideals.*

Now suppose that we take $\mathbb{Z}$. In this case the maximal ideals correspond to the prime numbers, and in addition there is one point whose closure is the whole spectrum. In this respect $\operatorname{Spec}\mathbb{Z}$ *is very similar to* $\operatorname{Spec}k[t]$.

We will need one very useful fact from commutative algebra:

Lemma 11.6. *If $\mathfrak{a}\trianglelefteq R$ is an ideal in a ring R then the radical of $\mathfrak{a}$ is the intersection of all prime ideals containing $\mathfrak{a}$.*

Proof. One inclusion is clear; every prime ideal $\mathfrak{p}$ is radical (that is equal to its own radical) and so the intersection of all prime ideals containing $\mathfrak{a}$ is radical.

Now suppose that r does not belong to the radical of $\mathfrak{a}$. Let $\mathfrak{b}$ be the ideal generated by the image of $\mathfrak{a}$ inside the ring R_r. Then the image of r inside the quotient ring $R_r/\mathfrak{b}$ is non-zero. Pick an ideal in this ring, maximal with respect to the property that it does not contain the image of r. Then the inverse image $\mathfrak{p}$ of this ideal is a prime ideal which does not contain r. $\square$

Lemma 11.7. *Let X be the spectrum of the ring R and let $f\in R$.*

If $U_f=\bigcup U_{g_i}$ then $f^n=\sum b_ig_i$, where $b_1,b_2,\dots,b_k\in R$. In particular U_f is compact.

Proof. Taking complements, we see that

$$V(\langle f\rangle)=\bigcap_i V(\langle g_i\rangle)=V(\langle\sum_i g_i\rangle).$$

Now $V(\mathfrak{a})$ consists of all prime ideals that contain $\mathfrak{a}$, and the radical of $\mathfrak{a}$ is the intersection of all the prime ideals that contain $\mathfrak{a}$. Thus

$$\sqrt{\langle f\rangle}=\sqrt{\langle\sum_i g_i\rangle}.$$

But then, in particular, f^n is a finite linear combination of the g_i and the corresponding open sets cover U_f. $\square$

As pointed out above, we need a slightly more general notion of a function than the one given above:

Definition 11.8. *Let R be a ring. We define a sheaf of rings $\mathcal{O}_X$ on the spectrum of R as follows. Let U be any open set of X. A section σ of $\mathcal{O}_X(U)$ is by definition any function*

$$s\colon U \longrightarrow \coprod_{\mathfrak{p}\in U} R_{\mathfrak{p}},$$

where $s(\mathfrak{p}) \in R_{\mathfrak{p}}$, which is locally represented by a quotient. More precisely, given a point $\mathfrak{q} \in U$, there is an element $f \in R$ such that $U_f \subset U$ and such that the section $\sigma|_U$ is represented by a/f^n, for some $a \in R$ and $n \in \mathbb{N}$.

An ***affine scheme*** *is then any locally ringed space isomorphic to the spectrum of a ring with its associated sheaf. A* ***scheme*** *is a locally ringed space, which is locally isomorphic, as locally ringed space, to an affine scheme.*

It is not hard to see that $\mathcal{O}_X(U)$ is a ring (sums and products are defined in the obvious way) and that we do in fact have a sheaf rather than just a presheaf.

The key result is the following:

Lemma 11.9. *Let X be an affine scheme, isomorphic to the the spectrum of R and let $f \in R$.*

(1) *For any $\mathfrak{p} \in X$, the stalk $\mathcal{O}_{X,\mathfrak{p}}$ is isomorphic to the local ring $R_{\mathfrak{p}}$.*
(2) *The ring $\mathcal{O}_X(U_f)$ is isomorphic to R_f.*

In particular $\mathcal{O}_X(X) \simeq R$.

Proof. We first prove (1). There is an obvious ring homomorphism

$$\mathcal{O}_{X,\mathfrak{p}} \longrightarrow R_{\mathfrak{p}},$$

which just sends a germ (g, U) to its value $g(\mathfrak{p})$ at $\mathfrak{p}$.

On the other hand, there is an obvious ring homomorphism,

$$R \longrightarrow \mathcal{O}_{X,\mathfrak{p}},$$

which sends an element $r \in R$ to the pair (r, X). Suppose that $f \notin \mathfrak{p}$. Then $(1/f, U_f)$ defines an element of $\mathcal{O}_{X,\mathfrak{p}}$, and this element is an inverse of (f, X). It follows, by the universal property of the localisation, that there is a ring homomorphism,

$$R_{\mathfrak{p}} \longrightarrow \mathcal{O}_{X,\mathfrak{p}},$$

which is the inverse map. Hence (1).

Now we turn to the proof of (2). As before there is an obvious ring homomorphism,

$$R \longrightarrow \mathcal{O}_X(U_f),$$

which induces a ring homomorphism

$$R_f \longrightarrow \mathcal{O}_X(U_f).$$

We have to show that this map is an isomorphism. We first consider injectivity. Suppose that $a/f^n \in R_f$ is sent to zero. Then for every $\mathfrak{p} \in \operatorname{Spec} R$, $f \notin \mathfrak{p}$, the image of a/f^n is equal to zero in $R_{\mathfrak{p}}$. For each such prime $\mathfrak{p}$ there is an element $h \notin \mathfrak{p}$ such that $ha = 0$ in R. Let $\mathfrak{a}$ be the annihilator of a in R. Then $h \in \mathfrak{a}$ and $h \notin \mathfrak{p}$, so that $\mathfrak{a}$ is not a subset of $\mathfrak{p}$. Since this holds for every $\mathfrak{p} \in U_f$, it follows that $V(\mathfrak{a}) \cap U_f = \emptyset$. But then $f \in \sqrt{\mathfrak{a}}$ so that $f^l \in \mathfrak{a}$, for some l. It follows that $f^l a = 0$, so that a/f^n is zero in R_f. Thus the map is injective.

Now consider surjectivity. Pick $s \in \mathcal{O}_X(U_f)$. By assumption, we may cover U_f by open sets V_i such that s is represented by $a_i/g_i^{n_i}$ on V_i. Replacing g_i by $g_i^{n_i}$ we may assume that $n_i = 1$. By definition $g_i \notin \mathfrak{p}$, for every $\mathfrak{p} \in V_i$, so that $V_i \subset U_{g_i}$. Now since sets of the form U_h form a base for the topology, we may assume that $V_i = U_{h_i}$. As $U_{h_i} \subset U_{g_i}$ it follows that $V(g_i) \subset V(h_i)$ so that

$$\sqrt{\langle h_i \rangle} \subset \sqrt{\langle g_i \rangle}.$$

But then $h_i^{n_i} \in \langle g_i \rangle$, so that $h_i^{n_i} = c_i g_i$. In particular

$$\frac{a_i}{g_i} = \frac{c_i a_i}{h_i^{n_i}}.$$

Replacing h_i by $h_i^{n_i}$ and a_i by $c_i a_i$, we may assume that U_f is covered by U_{h_i}, and that s is represented by a_i/h_i on U_{h_i}.

Now observe that by (11.7), $f^n = \sum b_i h_i$, where $b_1, b_2, \dots, b_k \in R$ and U_f can be covered by finitely many of the sets U_{h_i}. Thus we may assume that we have only finitely many h_i. Now on $U_{h_i h_j} = U_{h_i} \cap U_{h_j}$, there are two ways to represent s, one way by a_i/h_i and the other by a_j/h_j. By injectivity, we have $a_i/h_i = a_j/h_j$ in $R_{h_i h_j}$ so that for some n,

$$(h_i h_j)^n (h_j a_i - h_i a_j) = 0.$$

Since there are only finitely many i and j, we may assume that n is independent of i and j. We may rewrite this equation as

$$h_j^{n+1}(h_i^n a_i) - h_i^{n+1}(h_j^n a_j) = 0.$$

If we replace h_i by h_i^{n+1} and a_i by $h_i^n a_i$, then s is still represented by a_i/h_i and moreover

$$h_j a_i = h_i a_j.$$

Let $a = \sum_i b_i a_i$, where $f^n = \sum_i b_i h_i$. Then for each j,

$$\begin{aligned} h_j a &= \sum_i b_i a_i h_j \\ &= \sum_i b_i h_i a_j \\ &= f^n a_j. \end{aligned}$$

But then $a/f^n = a_j/h_j$ on U_{h_j}. But then a/f^n represents s on the whole of U_f. □

Note that by (2) of (11.9), we have achieved our aim of constructing a topological space from an arbitrary ring R, which realises R as a natural subset of the continuous functions.

Definition 11.10. *A morphism of schemes is simply a morphism between two locally ringed spaces which are schemes.*

The gives us a category, the category of schemes. Note that the category of schemes contains the category of affine schemes as a full subcategory and that the category of schemes is a full subcategory of the category of locally ringed spaces.

Theorem 11.11. *There is an equivalence of categories between the category of affine schemes and the category of commutative rings with unity.*

Proof. Let F be the functor that associates to an affine scheme, the global sections of the structure sheaf. Given a morphism

$$(f, f^{\#}) \colon (X, \mathcal{O}_X) = \operatorname{Spec} B \longrightarrow (Y, \mathcal{O}_Y) = \operatorname{Spec} A,$$

of locally ringed spaces then let

$$\phi \colon A \longrightarrow B,$$

be the induced map on global sections. It is clear that F is then a contravariant functor and F is essentially surjective by (11.9).

Now suppose that $\phi \colon A \longrightarrow B$ is a ring homomorphism. We are going to construct a morphism

$$(f, f^{\#}) \colon (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y),$$

of locally ringed spaces. Suppose that we are given $\mathfrak{p} \in X$. Then $\mathfrak{p}$ is a prime ideal of B. But then $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$ is a prime ideal of A. Thus we get a function $f : X \longrightarrow Y$. Now if $\mathfrak{a}$ is an ideal of A, then $f^{-1}(V(\mathfrak{a})) = V(\langle \phi(\mathfrak{a}) \rangle)$, so that f is certainly continuous. For each prime ideal $\mathfrak{p}$ of B, there is an induced morphism

$$\phi_{\mathfrak{p}} \colon A_{\phi^{-1}(\mathfrak{p})} \longrightarrow B_{\mathfrak{p}},$$

of local rings. Now suppose that $V \subset Y$ is an open set. We want to define a ring homomorphism

$$f^{\#}(V)\colon \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}(V)).$$

Suppose first that $V = U_g$, where $g \in A$. Then $\mathcal{O}_Y(V) = A_g$ and $f^{-1}(V) \subset U_{\phi(g)}$. But then there is a restriction map

$$\mathcal{O}_X(U_{\phi(g)}) \simeq B_{\phi(g)} \longrightarrow \mathcal{O}_X(f^{-1}(V)).$$

On the other hand, composing there is a ring homomorphism

$$A \longrightarrow B_{\phi(g)}.$$

Since the image of g is invertible, by the universal property of the localisation, there is an induced ring homomorphism

$$A_g \longrightarrow B_{\phi(g)}.$$

Putting all of this together, we have defined $f^{\#}(V)$ when $V = U_g$. Since the sets U_g form a base for the topology, and these maps are compatible in the obvious sense, this defines a morphism

$$f^{\#}\colon \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X,$$

of sheaves. Clearly the induced map on local rings is given by $\phi_{\mathfrak{p}}$, and so $(f, f^{\#})$ is a morphism of local rings.

Finally it suffices to prove that these two assignments are inverse. The composition one way is clear. If we start with ϕ and construct $(f, f^{\#})$ then we get back ϕ on global sections. Conversely suppose that we start with $(f, f^{\#})$, and let ϕ be the map on global sections. Given $\mathfrak{p} \in X$, we get a morphism of local rings on stalks, which is compatible with ϕ and localisation, so that we get a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\ A_{f(\mathfrak{p})} \arrow[r, "f^{\#}_{\mathfrak{p}}"] & B_{\mathfrak{p}}. \end{tikzcd}$$

But since $f^{\#}_{\mathfrak{p}}$ is a morphism of local rings, it follows that $\phi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$, so that f coincides with the map induced by ϕ. But then $f^{\#}$ is also the map induced by ϕ. □

Definition 11.12. *Let X be a scheme and let $x \in X$ be a point of X. The **residue field of** X **at** x is the quotient of $\mathcal{O}_{X,x}$ by its maximal ideal.*

We recall some basic facts about valuations and valuation rings.

Definition 11.13. *Let K be a field and let G be a totally ordered abelian group. A* ***valuation*** *of K with values in G, is a map*

$$\nu\colon K - \{0\} \longrightarrow G,$$

such that for all x and $y \in K - \{0\}$ we have:

(1) $\nu(xy) = \nu(x) + \nu(y)$.
(2) $\nu(x + y) \geq \min(\nu(x), \nu(y))$.

Definition-Lemma 11.14. *If ν is a valuation, then the set*

$$R = \{\, x \in K \mid \nu(x) \geq 0 \,\} \cup \{0\},$$

is a subring of K, which is called the ***valuation ring*** *of ν. The set*

$$\mathfrak{m} = \{\, x \in K \mid \nu(x) > 0 \,\} \cup \{0\},$$

is an ideal in R and the pair $(R, \mathfrak{m})$ is a local ring.

Proof. Easy check. □

Definition 11.15. *A valuation is called a* ***discrete valuation*** *if $G = \mathbb{Z}$ and ν is surjective. The corresponding valuation ring is called a* ***discrete valuation ring****. Any element $t \in R$ such that $\nu(t) = 1$ is called a* ***uniformising parameter****.*

Lemma 11.16. *Let R be an integral domain, which is not a field.*

The following are equivalent:

- *R is a DVR.*
- *R is a local ring and a PID.*

Proof. Suppose that R is a DVR. Then R is certainly a local ring. Suppose that a and $b \in R$ and $\nu(a) = \nu(b)$. Then $\nu(b/a) = \nu(b) - \nu(a) = 0$ and so $\langle a \rangle = \langle b \rangle$. It follows that the ideals of R are of the form

$$I_k = \{\, a \in R \mid \nu(a) \geq k \,\}.$$

As ν is surjective, there is an element $t \in R$ such that $\nu(t) = 1$. Then

$$I_k = \langle t^k \rangle = \mathfrak{m}^k.$$

Thus R is a PID.

Now suppose that R is a local ring and a PID. Let $\mathfrak{m}$ be the unique maximal ideal. As R is a PID, $\mathfrak{m} = \langle t \rangle$, for some $t \in R$. Define a map

$$\nu\colon K \longrightarrow \mathbb{Z},$$

by sending a to k, where $a \in \mathfrak{m}^k - \mathfrak{m}^{k+1}$ and extending this to any fraction a/b in the obvious way. It is easy to check that ν is a valuation and that R is the valuation ring. □

There are two key examples of a DVR. First let k be field and let $R = k[t]_{\langle t\rangle}$. Then R is a local ring and a PID so that R is a DVR. t is a uniformising parameter. Note that R is the stalk of the struture shear of the affine line at the origin.

Now let

$$\Delta = \{\, z \in \mathbb{C} \,|\, |z| < 1 \,\},$$

be the unit disc in the complex plane. Then the stalk $\mathcal{O}_{\Delta,0}$ of the sheaf of holomorphic functions is a local ring. The order of vanishing realises this ring as a DVR. z is a uniformising parameter.

In fact if C is a smooth algebraic curve, an algebraic variety of dimension one, then $\mathcal{O}_{C,p}$ is a DVR.

Example 11.17. *Let R be the local ring of a curve over an algebraically closed field (or more generally a discrete valuation ring). Then* $\operatorname{Spec} R$ *consists of two points; the maximal ideal, and the zero ideal. The first t_0 is closed and has residue field the groundfield k of C, the second t_0 has residue field the quotient ring K of R, and its closure is the whole of X. The inclusion map $R \longrightarrow K$ corresponds to a morphism which sends the unique point of* $\operatorname{Spec} K$ *to t_1.*

There is another morphism of ringed spaces which sends the unique point of $\operatorname{Spec} K$ *to t_0 and uses the inclusion above to define the map on structure sheaves. Since there is only one way to map R to K, this does not come from a map on rings. In fact the second map is not a morphism of locally ringed spaces, and so it is not a morphism of schemes.*

Example 11.18. *It is interesting to see an example of an affine scheme, in a seemingly esoteric case. Consider the case of a number field k (that is a finite extension of $\mathbb{Q}$, with its ring of integers $A \subset k$ (that is the integral closure of $\mathbb{Z}$ inside k). As a particular example, take $k = \mathbb{Q}(\sqrt{3})$. Then $A = \mathbb{Z} \oplus \mathbb{Z}\langle\sqrt{3}\rangle$. The picture is very similar to the case of $\mathbb{Z}$. There are infinitely many maximal ideals, and only one point which is not closed, the zero ideal. Moreover, as there is a natural ring homomorphism $\mathbb{Z} \longrightarrow A$, by our equivalence of categories, there is an induced morphism of schemes* $\operatorname{Spec} A \longrightarrow \operatorname{Spec} \mathbb{Z}$. *We investigate this map. Consider the fibre over a point $\langle p\rangle \in \operatorname{Spec} \mathbb{Z}$. This is just the set of primes in A containing the ideal pA. It is well known by number theorists, that three things can happen:*

1. *If p divides the discriminant of $k/\mathbb{Q}$ (which in this case is* 12*), that is $p = 2$ or* 3*, then the ideal $\langle p\rangle$ is a square in A.*

$$\langle 2\rangle A = \langle -1 + \sqrt{3}\rangle^2,$$

and
$$\langle 3\rangle A = \langle\sqrt{3}\rangle^2.$$

(2) *If 3 is a square modulo p, the prime $\langle p\rangle$ factors into a product of distinct primes,*
$$\langle 11\rangle A = \langle 4+3\sqrt{3}\rangle\langle 4-3\sqrt{3}\rangle,$$
or
$$\langle 13\rangle A = \langle 4+\sqrt{3}\rangle\langle 4-\sqrt{3}\rangle,$$

(3) *If $p > 3$ and 3 is not a square mod p (e.g $p = 5$ and 7), the ideal $\langle p\rangle$ is prime in A.*

Definition 11.19. *Let $\mathcal{C}$ be a category and let X be an object of $\mathcal{C}$. Let $\mathcal{D} = \mathcal{C}|_X$ be the category whose objects consist of pairs $f\colon Y \longrightarrow X$, where f is a morphism of $\mathcal{C}$, and whose morphisms, consist of commutative diagrams*

$$\begin{tikzcd} Y \arrow[rr, "\phi"] \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & X. & \end{tikzcd}$$

$\mathcal{D}$ is known as the category over X. If X is a scheme, then a scheme over X is exactly an object of the category of schemes over X. Let R be a ring. ***Affine n-space over*** *R, denoted $\mathbb{A}^n_R$, is the spectrum of the polynomial ring $R[x_1, x_2, \dots, x_n]$.*

One of the key ideas of schemes, is to work over arbitrary bases. Note that since there is an inclusion $R \longrightarrow R[x_1, x_2, \dots, x_n]$ of rings, affine space over R is a scheme over $\operatorname{Spec} R$. Thus we may define affine space over any affine scheme.