Let $X$ be a topological space (respectively ringed space). The category of presheaves of groups (respectively $\mathcal{O}_X$-modules) on $X$ is denoted $\mathcal{P}$ and the category of sheaves of groups (respectively $\mathcal{O}_X$-modules) on $X$ is denoted $\mathcal{S}$.

1. A **zero object** of a category $\mathcal{C}$ is any object which is simultaneously an initial and a terminal object. Identify a zero object in $\mathcal{P}$.

2. If $\mathcal{C}$ is a category with a zero object $0$ and $X$ and $Y$ are two objects of $\mathcal{C}$, the **zero morphism** $0_{XY} : X \to Y$ is the composition of $X \to 0$ and $0 \to Y$. What is the zero morphism between two presheaves $F$ and $G$ on $X$?

3. If $\mathcal{C}$ is a category with zero object $0$ and $f : X \to Y$ is a morphism in $\mathcal{C}$ then the kernel of $f$ is the equaliser of $f$ and the zero morphism $0_{XY}$. If $f : F \to G$ is a morphism of presheaves on the topological space $X$, then show that the presheaf $\text{Ker} \ f$, defined by
   \[ U \to \text{Ker} \ f(U), \]
   is a kernel in the category $\mathcal{P}$. If $F$ and $G$ are sheaves (ie if $f$ is a morphism in $\mathcal{S}$) then show that $\text{Ker} \ f$ is also a kernel in the category $\mathcal{S}$.

4. Let $I$ be the category consisting of two objects and four morphisms, the two identity maps and two morphisms going from the first object to the second. If $F : I \to \mathcal{C}$ is any functor, the direct limit is called the co-equaliser (so that the co-equaliser is the dual notion of the equaliser). If $\mathcal{C}$ is a category with zero object $0$ and $f : X \to Y$ is a morphism in $\mathcal{C}$ then the cokernel of $f$ is the co-equaliser of $f$ and the zero morphism $0_{XY}$.
   (i) Identify the cokernel of a morphism of presheaves $f : F \to G$.
   (ii) Identify the cokernel of a morphism of sheaves $f : F \to G$.
   (iii) Give examples to show that if $f : F \to G$ is a morphism of sheaves, we get different answers if we take the cokernel in $\mathcal{P}$ or in $\mathcal{S}$.

5. Suppose we are given morphisms of sheaves $f_i : \mathcal{F}_i \to \mathcal{F}_{i+1}$. We say that this sequence is exact at $\mathcal{F}_i$, if $\text{Ker} \ f_i = \text{Im} \ f_{i-1}$ (as subsheaves of $\mathcal{F}_i$).
   (i) Show that
   \[ 0 \to \mathcal{F} \to \mathcal{G}, \]
   is exact at $\mathcal{F}$ iff $\mathcal{F} \to \mathcal{G}$ is injective.
(ii) Show that
\[ \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0, \]
is exact at \( \mathcal{G} \) iff \( \mathcal{F} \rightarrow \mathcal{G} \) is surjective.

(iii) Show that the sequence above is exact at \( \mathcal{F} \) iff the induced sequence of maps of stalks is exact at \( \mathcal{F}_{ip} \) for every \( p \in X \).

(iv) Let
\[ 0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H}, \]
be an exact sequence of sheaves on the topological space \( X \). Show that if \( U \subset X \) is any open set then the sequence
\[ 0 \rightarrow \mathcal{F}(U) \rightarrow \mathcal{G}(U) \rightarrow \mathcal{H}(U), \]
is exact.

(v) Give examples to show that a similar result fails for short exact sequences
\[ 0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0. \]