HWK #5, DUE WEDNESDAY 10/14

Let X be a topological space (respectively ringed space). The category of presheaves of groups (respectively $\mathcal{O}_X$-modules) on X is denoted $\mathcal{P}$ and the category of sheaves of groups (respectively $\mathcal{O}_X$-modules) on X is denoted $\mathcal{S}$.

1. A **zero object** of a category $\mathcal{C}$ is any object which is simultaneously an initial and a terminal object. Identify a zero object in $\mathcal{P}$.

2. If $\mathcal{C}$ is a category with a zero object 0 and X and Y are two objects of $\mathcal{C}$, the **zero morphism** $0_{XY} \colon X \longrightarrow Y$ is the composition of $X \longrightarrow 0$ and $0 \longrightarrow Y$. What is the zero morphism between two presheaves $\mathcal{F}$ and $\mathcal{G}$ on X?

3. If $\mathcal{C}$ is a category with zero object 0 and $f \colon X \longrightarrow Y$ is a morphism in $\mathcal{C}$ then the kernel of f is the equaliser of f and the zero morphism 0_{XY}. If $f \colon \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of presheaves on the topological space X, then show that the presheaf $\operatorname{Ker} f$, defined by

$$U \longrightarrow \operatorname{Ker} f(U),$$

is a kernel in the category $\mathcal{P}$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves (ie if f is a morphism in $\mathcal{S}$) then show that $\operatorname{Ker} f$ is also a kernel in the category $\mathcal{S}$.

4. Let $\mathbb{I}$ be the category consisting of two objects and four morphisms, the two identity maps and two morphisms going from the first object to the second. If $F \colon \mathbb{I} \longrightarrow \mathcal{C}$ is any functor, the direct limit is called the co-equaliser (so that the co-equaliser is the dual notion of the equaliser) If $\mathcal{C}$ is a category with zero object 0 and $f \colon X \longrightarrow Y$ is a morphism in $\mathcal{C}$ then the cokernel of f is the co-equaliser of f and the zero morphism 0_{XY}.

(i) Identify the cokernel of a morphism of presheaves $f \colon \mathcal{F} \longrightarrow \mathcal{G}$.
(ii) Identify the cokernel of a morphism of sheaves $f \colon \mathcal{F} \longrightarrow \mathcal{G}$.
(iii) Give examples to show that if $f \colon \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of sheaves, we get different answers if we take the cokernel in $\mathcal{P}$ or in $\mathcal{S}$.

5. Suppose we are given morphisms of sheaves $f_i \colon \mathcal{F}_i \longrightarrow \mathcal{F}_{i+1}$. We say that this sequence is exact at $\mathcal{F}_i$, if $\operatorname{Ker} f_i = \operatorname{Im} f_{i-1}$ (as subsheaves of $\mathcal{F}_i$).

(i) Show that

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G},$$

is exact at $\mathcal{F}$ iff $\mathcal{F} \longrightarrow \mathcal{G}$ is injective.

(ii) Show that

$$\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0,$$

is exact at $\mathcal{G}$ iff $\mathcal{F} \longrightarrow \mathcal{G}$ is surjective.

(iii) Show that the sequence above is exact at $\mathcal{F}_i$ iff the induced sequence of maps of stalks is exact at $\mathcal{F}_{ip}$ for every $p \in X$.

(iv) Let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H},$$

be an exact sequence of sheaves on the topological space X. Show that if $U \subset X$ is any open set then the sequence

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U),$$

is exact.

(v) Give examples to show that a similar result fails for short exact sequences

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$