HWK #3, DUE WEDNESDAY 09/30

1. A **groupoid** is a category in which every morphism is an isomorphism. Show that one could define a group as a small groupoid with one object.
2. A **monomorphism** in a category $\mathcal{C}$ is a morphism $f\colon X \longrightarrow Y$ such that if $g_i\colon Z \longrightarrow X$ are two morphisms, $i = 1, 2$ and $f \circ g_1 = f \circ g_2$, then in fact $g_1 = g_2$ (so that monomorphisms are to morphisms as injective functions are to functions).
If $f\colon X \longrightarrow Y$ is a monomorphism, then show that for every pair of morphisms $W \longrightarrow X$ and $Z \longrightarrow X$, $W \underset{X}{\times} Z$ and $W \underset{Y}{\times} Z$ are isomorphic.
3. Let $\mathbb{I}$ be the category consisting of two objects and four morphisms, the two identity maps and two morphisms going from the first object to the second.
(i) Show that to give a functor $F\colon \mathbb{I} \longrightarrow \mathcal{C}$ is the same as to pick two objects X and Y of $\mathcal{C}$ and two morphisms $f_i\colon X \longrightarrow Y$, $i = 1$ and 2.
(ii) The equaliser of f_1 and f_2 is the limit of the corresponding functor. Identify the equaliser in the category of sets.
(iii) Let $\mathcal{C}$ be a category which admits products. Show that $\mathcal{C}$ admits fibre products if and only if it admits equalisers.
(iv) Give an example of a category which admits fibre products but not equalisers.
4. (i) Given two categories $\mathcal{C}$ (which we assume is locally small, for technical reasons) and $\mathcal{D}$, show that one can form a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, whose objects are the functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural transformations between functors
(ii) Let $\mathbb{I}$ and $\mathcal{C}$ be two categories, where $\mathcal{C}$ admits all limits of functors $F\colon \mathbb{I} \longrightarrow \mathcal{C}$. Show that there is a functor

$$\varprojlim_{\mathbb{I}}\colon \mathrm{Fun}(\mathbb{I}, \mathcal{C}) \longrightarrow \mathcal{C},$$

which assigns to every functor $F\colon \mathbb{I} \longrightarrow \mathcal{C}$, the limit $\varprojlim_{\mathbb{I}} F$. In particular, given a natural transformation $\alpha\colon F \longrightarrow G$, how should one define $\varprojlim_{\mathbb{I}}(\alpha)$?
(iii) Suppose that we have morphisms $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ and $h\colon W \longrightarrow Z$, in a category $\mathcal{C}$ which admits fibre products. Show that there is a natural isomorphism between $X \underset{Z}{\times} W$ and $X \underset{Y}{\times} (Y \underset{Z}{\times} W)$.

5. Prove Yoneda's Lemma.