1. A **groupoid** is a category in which every morphism is an isomorphism. Show that one could define a group as a small groupoid with one object.

2. A **monomorphism** in a category $C$ is a morphism $f : X \rightarrow Y$ such that if $g_i : Z \rightarrow X$ are two morphisms, $i = 1, 2$ and $f \circ g_1 = f \circ g_2$, then in fact $g_1 = g_2$ (so that monomorphisms are to morphisms as injective functions are to functions).

   If $f : X \rightarrow Y$ is a monomorphism, then show that for every pair of morphisms $W \rightarrow X$ and $Z \rightarrow X$, $W \times X$ and $W \times Z$ are isomorphic.

3. Let $I$ be the category consisting of two objects and four morphisms, the two identity maps and two morphisms going from the first object to the second.

   (i) Show that to give a functor $F : I \rightarrow C$ is the same as to pick two objects $X$ and $Y$ of $C$ and two morphisms $f_i : X \rightarrow Y$, $i = 1$ and 2.

   (ii) The equaliser of $f_1$ and $f_2$ is the limit of the corresponding functor. Identify the equaliser in the category of sets.

   (iii) Let $C$ be a category which admits products. Show that $C$ admits fibre products if and only if it admits equalisers.

   (iv) Give an example of a category which admits fibre products but not equalisers.

4. (i) Given two categories $C$ (which we assume is locally small, for technical reasons) and $D$, show that one can form a category $\text{Fun}(C, D)$, whose objects are the functors from $C$ to $D$ and whose morphisms are natural transformations between functors.

   (ii) Let $I$ and $C$ be two categories, where $C$ admits all limits of functors $F : I \rightarrow C$. Show that there is a functor $\lim_i : \text{Fun}(I, C) \rightarrow C$, which assigns to every functor $F : I \rightarrow C$, the limit $\lim_i F$. In particular, given a natural transformation $\alpha : F \rightarrow G$, how should one define $\lim_i (\alpha)$?

   (iii) Suppose that we have morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ and $h : W \rightarrow Z$, in a category $C$ which admits fibre products. Show that there is a natural isomorphism between $X \times W$ and $X \times (Y \times W)$. 

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5. Prove Yoneda’s Lemma.