HWK #2, DUE WEDNESDAY 09/23

1. Let \( p_1, p_2, \ldots, p_{n+2} \) and \( q_1, q_2, \ldots, q_{n+2} \) be two sets of \( n+2 \) points in linear general position in \( \mathbb{P}^n \). Show that there is a unique element of \( \text{PGL}(n+1) \) sending \( p_i \) to \( q_i \).

2. Let \( K \) be an algebraically closed field. Show that, up to conjugacy, any element \( \phi \) of \( \text{PGL}(2, K) \) is one of
   
   (1) the identity,
   (2) \( z \rightarrow az, \ a \in K^* \),
   (3) \( z \rightarrow z + 1 \),

   and that the three cases are distinguished by the number of fixed points; at least three; two; one.

3. Show that the twisted cubic is defined by the equations \( XW = YZ \), \( Y^2 = XZ \) and \( Z^2 = YW \).

4. a) Show the intersection of any two of the quadrics above is the union of \( C \) and a line (in fact either a tangent line or a secant line, that is a line which meets \( C \) twice).
   
b) More generally, if \( \lambda = [\lambda_0 : \lambda_1 : \lambda_2] \) is a point of \( \mathbb{P}^2 \), let \( F_\lambda \) denote the quadratic polynomial
   
   \[ \lambda_0(Y^2 - XZ) + \lambda_1(XW - YZ) + \lambda_2(Z^2 - YW). \]

   Show that if \( \lambda \neq \mu \) then the zero locus of \( F_\lambda \) and \( F_\mu \) is also the union of \( C \) and a line (again, in fact either a tangent or secant line).

5. Show that any set of points on a rational normal curve are in linear general position.

6. Show that the image of \( \mathbb{P}^n \) under the \( d \)-uple embedding is defined by the equations
   
   \[ Z_I Z_J = Z_{I'} Z_{J'}, \]

   where \( I, J, I' \) and \( J' \) are any \((n+1)\)-tuples of positive integers who sum is \( d \) and

   \[ I + J = I' + J'. \]