MODEL ANSWERS TO THE SIXTH HOMEWORK

1. A dense open subset is given by taking l distinct reduced points. For each point we have ∞^n choices (any point of X) and so the dimension is nl.

2. (i) Let $R = k[x_1, x_2, \dots, x_n]$. Then

$$\dim_k \frac{R}{\mathfrak{m}^p},$$

is the number of monomials of degree at most $p-1$ in n variables. This is the same as the number of monomials of degree $p-1$ in $n+1$ variables. The usual stars and bars argument says that this is

$$\binom{n+p-1}{n}.$$

So the length of z is

$$l = \binom{n+p-1}{n} + q.$$

(ii) Clear, since we just define the product of any element of W with any homogeneous element of degree at least one to be zero.

(iii) The dimension of V is equal to

$$m = \binom{n+p-2}{n-1},$$

the number of monomials of degree $p-1$ in n variables. The dimension of the Grassmannian of q planes in the vector space of planes in V is then $q(m-q)$. Thus the space of such zero dimensional schemes has dimension

$$n + q(m-q),$$

since we also get to choose the support of z.

3. Let us take $n = 3$. Fixing m, we want to maximise the product

$$3 + q(m-q).$$

Clearly we should take $q = \llcorner m/2 \lrcorner$. In order to simplify the notation, we will tacitly assume that m is even, so that $q = m/2$. Then the dimension of the space of curvilinear schemes is

$$3l = 3\binom{2+p}{3} + 3/2\binom{p}{2},$$

and the dimension of the space of schemes from (ii) is

$$3 + 1/4\binom{p}{2}^2.$$

Since the first expression is a polynomial of degree 3 in p and the second expression is a polynomial of degree 4 in p, if p is large enough then we win. It is still interesting to figure out exactly when this happens. In fact $p = 10$ will do, in which case the length is 138. In $\mathbb{A}^4_k$, $p = 5$ will do.