MODEL ANSWERS TO THE FOURTH HOMEWORK

We will need:
2.17 (a) The map on topological spaces is surely a homeomorphism under these circumstances. It suffices then to check that the map on structure sheaves is an isomorphism. As this may be checked on stalks, the result follows.
(b) If \( X \) is affine, just take \( r = f = 1 \).
Otherwise suppose that we have \( f_1, f_2, \ldots, f_r \) such that \( U_{f_i} \) is affine, where \( f_1, f_2, \ldots, f_r \) generate the unit ideal. Let \( Y = \text{Spec} \ A \). By (2.4) there is a morphism

\[
f : X \longrightarrow Y.
\]

This morphism is certainly an isomorphism over the open subset \( U_{f_i} \), by (2.16.d). But since \( f_1, f_2, \ldots, f_r \) generate the unit ideal, these sets cover \( X \) and we are done by (a).
3.1 Suppose that \( V = \text{Spec} \ B \) is an affine open subset of \( Y \), such that \( f^{-1}(V) \) can be covered by open affine subsets \( U_j = \text{Spec} \ A_j \), where each \( A_j \) is a finitely generated \( B \)-algebra. Let \( g \in B \) and consider \( V_g = \text{Spec} \ B_g \). If \( h_j \) denotes the image of \( g \) in \( A_j \), it follows that \( f^{-1}(V_g) \) is covered by open affine subsets of the form \( \text{Spec}(A_j)_{h_j} \). But as \( A_j \) is a finitely generated \( B \)-algebra, certainly \( (A_j)_{h_j} \) is a finitely generated \( B_g \)-algebra.
As open sets of this form cover \( V \), it follows that we can cover \( V \) by open affine subsets whose inverse image is a union of affine open subsets, which are the spectra of finitely generated algebras. Renaming, we might as well assume that \( V = X \) is affine. Arguing as in lectures, we may assume that the open affine cover of \( X \) is given by basic open sets. Since \( X \) is affine, we may assume that we have finitely many of these basic open sets. In other words, we are given \( f_1, f_2, \ldots, f_k \), which generate the unit ideal of \( B \) and the inverse image of \( V_i = U_{f_i} \) is covered by open affines which are spectra of finitely generated \( B_{f_i} \)-algebras. For each \( i \), pick an open set \( U_i = \text{Spec} \ A_i \), in the inverse image \( f^{-1}(V_i) \), where \( A_i \) is a finitely generated \( B_{f_i} \)-algebra.
Let \( U \) be the union of the \( U_i \). Let \( g_i \) be the image of \( f_i \) in \( A = \Gamma(U, \mathcal{O}_U) \). Then \( U_i \) is the locus where \( g_i \) is not equal to zero. \( g_1, g_2, \ldots, g_k \) generate the unit ideal of \( A \), as \( f_1, f_2, \ldots, f_k \) generate the unit ideal of \( B \). It follows by (2.17), that \( U \) is affine.
So now we are reduced to the following problem in algebra. Let $B$ be an $A$-algebra, and let $f_1, f_2, \ldots, f_k$ generate the unit ideal. Suppose that $g_i$ is the image of $f_i$ and suppose that $A_i = A_{g_i}$ is a finitely generated $B_i = B_{f_i}$-algebra. Then $A$ is a finitely generated $B$-algebra.

To this end, pick generators $c_{i1}, c_{i2}, \ldots, c_{il_i}$ of $A_i$ over $B_i$. Then each $c_{ij}$ has the form $a_{ij}/g_i^n$, where we may assume that $n$ is constant, as we have only finitely many indices. I claim that $a_{ij}$, for every $i$ and $j$, generates $A$ over $B$. Pick $a \in A$. Then if $\phi_i: A \rightarrow A_i$ is the natural map, we have

$$\phi_i(a) = p(c_{ij}),$$

for some polynomial $p$, with coefficients in $B_i$. Clearing denominators, we then have

$$g_i^N a = q(a_{ij}),$$

for some polynomial $q$, with coefficients in $A_i$. We may write

$$\sum h_i g_i^N = 1,$$

for some $h_i$. But then

$$a = \sum h_i g_i^N a,$$

$$= \sum h_i \left( \sum q(a_{ij}) \right),$$

as required.

3.2 The key observation is that a scheme is compact iff it is the finite union of affine subschemes. Indeed, if $X$ is a scheme, then it is union of open affine subschemes, and if $X$ is compact, then finitely many cover. Conversely, any affine scheme is compact, and the finite union of compact sets is always compact.

So now suppose that $f: X \rightarrow Y$ is a compact morphism, and let $V$ be an affine subset. Let $W = \text{Spec } A$ be any open subset affine such that $f^{-1}(W)$ is compact, and let $U_f$ be an open affine subset of $W$, where $f \in A$. By assumption $f^{-1}(W)$ has an affine cover $W_i$. But then $W_i \cap U_f$ is a basic open affine subset, equal to $U_g$, where $g$ is the image of $f$. Since these sets cover $f^{-1}(W \cap U_f)$, it follows that $f^{-1}(W \cap U_f)$ is compact. Since any open affine subset $U$ of $W$ is the finite union of basic open subsets, it follows that if $f^{-1}(U)$ is compact.

It follows that we may refine our cover so that $V$ is covered by open affines, whose inverse images are compact. As $V$ is compact, we may take a finite subcover, and so $f^{-1}(V)$ is compact.
3.3 (a) Clear, from the first paragraph of 3.2.
(b) Simply apply 3.1 and 3.2.
(c) By now standard tricks, we can reduce this problem to showing that if an \( A \)-algebra \( B \) contains elements \( f_1, f_2, \ldots, f_k \) which generate the unit ideal and \( Bf_i \) is a finitely generated \( A \)-algebra, then so is \( B \). But this is proved in an almost identical fashion to the proof of 3.1.

3.4 Follows almost exactly the same proof as 3.1, and 3.3 (c). Replace the polynomial \( p \) by \( c_{ij} \) and the polynomial \( q \) by \( a_{ij} \).

3.6 Let \( U = \text{Spec } A \) be any open affine subscheme. Then \( \xi \in U \) and so \( \xi \) corresponds to a prime ideal of \( A \), which must be the zero ideal, or else \( \xi \) would not the generic point. But then

\[
O_{X, \xi} \cong A_{(0)} = K,
\]

where \( K \) is the field of fractions of \( A \).

3.8 We have to check the patching condition. Suppose that \( U \) and \( V \) are two affine open subschemes of \( X \). Let \( \bar{U} = \text{Spec } \bar{A} \) and \( \bar{V} = \text{Spec } \bar{B} \).

We have to exhibit a canonical isomorphism

\[
\phi: \bar{U} \longrightarrow \bar{V},
\]

where \( \bar{U} \) is the inverse image of \( U \cap V \) in \( \bar{U} \) and \( \bar{V} \) is the inverse image of \( U \cap V \) in \( \bar{V} \).

Since it suffices to construct a canonical morphism on an open cover, we may assume that \( U \) and \( V \) are open affines of a common affine scheme \( W = \text{Spec } C \) and that \( A = C_f \) and \( B = C_g \), where \( f \) and \( g \) belong to \( C \). It suffices to check that if \( \bar{A} \) is the integral closure of \( A \), then \( \bar{A}_f \) is the integral closure of \( A_f \). It is clear that any element of \( \bar{A}_f \) is integral over \( A_f \). Indeed if \( a/f^k \in \bar{A}_f \), where \( a \in \bar{A} \) satisfies the monic polynomial

\[
x^n + a_{n-1}x^{n-1} + \ldots + a_0,
\]

then \( a/f^k \) satisfies the monic polynomial

\[
x^n + b_{n-1}x^{n-1} + \ldots + b_0,
\]

where \( b_i = a_i/f^{n(k-i)} \). On the other hand if \( u \) belong to the integral closure of \( A_f \), then \( u \) is a root of a monic polynomial

\[
x^n + b_{n-1}x^{n-1} + \ldots + b_0,
\]

where each \( b_i \in A_f \). Clearing denominators, it follows that \( a = f^i u \in \bar{A} \), for an appropriate power of \( f \). Thus one can glue the schemes \( \bar{U} \) together to get a scheme \( \bar{X} \). The inclusion \( A \longrightarrow \bar{A} \) induces a morphism of schemes \( \bar{U} \longrightarrow \bar{U} \), whence a morphism of schemes \( \bar{U} \longrightarrow X \). Arguing as before, these morphisms agree on overlaps. It follows that there is an induced morphism \( \bar{X} \longrightarrow X \).
Now suppose that there is a dominant morphism of schemes \(Z \to X\), where \(Z\) is normal. This induces a dominant morphism \(Z_U \to U\), where \(U\) is an open affine subscheme and \(Z_U\) is the inverse image of \(U\). Thus it suffices to prove the universal property of \(X\) in the case when \(X\) is affine. Covering \(Z\) by open affines, it suffices to prove this result when \(Z\) is affine. Using the equivalence of categories, we are reduced to proving that if \(A \to \tilde{A}\) is the inclusion of \(A\) inside its integral closure, and \(A \to B\) is a ring homomorphism, where \(B\) is integrally closed, then there is a morphism \(\tilde{A} \to B\). Clearly there is such a morphism into the field of fractions \(L\) of \(B\). On the other hand, any element of the image is obviously integral over the image of \(A\), and so integral over \(B\). But then the image of \(\tilde{A}\) lies in \(B\), as \(B\) is integrally closed.

Suppose that \(X\) is of finite type. Clearly we may assume that \(X = \text{Spec } A\) is affine. We are reducing to showing that the integral closure \(\tilde{A}\) of a finitely generated \(k\)-algebra \(A\), is a finitely generated \(A\)-module. But this is a well-known result in algebra.

3.9 (a) 
\[
\mathbb{A}^2_k = \text{Spec } k[x, y] = \text{Spec}(k[x] \otimes_k k[y]) = \mathbb{A}^1_k \times \mathbb{A}^1_k.
\]

The points of \(\mathbb{A}^1_k\) consist of the maximal ideals \(m_a\) and the generic point \(\xi\). The points of the product of sets are then ordered pairs \((m_a, m_b)\), with closure \(\{(m_a, m_b)\} \cup \{(m_a, \xi)\}\), \((\xi, m_b)\) with closure 
\[
\{(m_a, m_b) \mid b \in k\} \cup \{(\xi, m_b)\},
\]
and \((\xi, \xi)\), whose closure is the whole space. Let \(\eta = (xy - 1)\). Then \(\eta\) is a prime ideal, whose closure is the set
\[
\{(m_a, m_b) \mid ab = 1\} \cup \{\eta\}.
\]

Thus \(\eta\) is not a point of the product of the two sets.

(b) As a topological space, \(X = \text{Spec } k(s) \times_k k(t)\) contains many points; \(k(s) \times_k k(t)\) is the localisation of \(k[s, t]\) of the multiplicative set \(S\) generated by the irreducible polynomials in \(s\) and \(t\). But this leaves many irreducible polynomials in both \(s\) and \(t\) which are not inverted, and each of these will generate a prime ideal. In fact \(X\) bears almost no relation to \(\mathbb{A}^2_k\).