MODEL ANSWERS TO THE THIRD HOMEWORK

2.1 By the universal property of the localisation there is a ring homomorphism

$$A \longrightarrow A_f.$$

Since we have an equivalence of categories, this induces a morphism of schemes

$$(g, g^{\#})\colon \operatorname{Spec} A_f \longrightarrow \operatorname{Spec} A = (X, \mathcal{O}_X).$$

Given $\mathfrak{q} \triangleleft A_f$, a prime ideal of A_f, $g(\mathfrak{q}) = \mathfrak{q} \cap A$ is a prime ideal of A which does not contain f. Thus we get an induced morphism

$$(g, g^{\#})\colon \operatorname{Spec} A_f \longrightarrow (U_f, \mathcal{O}_{U_f} = \mathcal{O}_X|_{U_f}),$$

and it suffices to prove that this morphism is an isomorphism.
We first show that g is a homeomorphism. Now a prime ideal of $\mathfrak{q} \triangleleft A_f$ gives rise to a surjective ring homomorphism $A_f \longrightarrow B$, where $B = A/\mathfrak{q}$ is an integral domain. Composing, we get a surjective ring homomorphism $A \longrightarrow B$, and the kernel is a prime ideal $\mathfrak{p} = g(\mathfrak{q}) = \mathfrak{q} \cap A$, which does not contain f. Conversely, a prime ideal of $\mathfrak{p} \triangleleft A$ not containing f gives rise to a surjective ring homomorphism $A \longrightarrow B$, where B is an integral domain and the image f' of f is not zero. Composing with the localisation map $B \longrightarrow B_{f'}$ we get a ring homomorphism with the same kernel, and the image of f is invertible. This gives us a surjective ring homomorphism $A_f \longrightarrow B_{f'}$ by the universal property of the localisation, and the kernel $\mathfrak{q} = \mathfrak{p}A_f$ is a prime ideal of A_f. It follows that g is a bijection.
On the other hand

$$\begin{aligned} \mathfrak{p} \in V(\mathfrak{a}) &\Leftrightarrow \mathfrak{a} \subset \mathfrak{p} \\ &\Leftrightarrow \mathfrak{a}A_f \subset \mathfrak{p}A_f \\ &\Leftrightarrow \mathfrak{p}A_f \in V(\mathfrak{a}A_f). \end{aligned}$$

Thus g is a homeomorphism.
To see that $g^{\#}$ is an isomorphism, it suffices to check that it is an isomorphism stalk by stalk. If $\mathfrak{p} \triangleleft A$ is any prime ideal and $a \notin \mathfrak{p}$ then the image of a in $(A_f)_{\mathfrak{q}}$ is a unit, where $\mathfrak{q} = g(\mathfrak{p})$. It follows that there is a natural ring homomorphism

$$g^{\#}_{\mathfrak{p}}\colon \mathcal{O}_{X,\mathfrak{p}} \simeq A_{\mathfrak{p}} \longrightarrow (A_f)_{\mathfrak{q}} \simeq \mathcal{O}_{U_f,\mathfrak{q}},$$

Composing, we get a ring homomorphism $A \longrightarrow (A_f)_{g(\mathfrak{p})}$, which satisfies the same universal property as $A \longrightarrow A_{\mathfrak{p}}$, so that it is an isomorphism

2.2 The pair $(U, \mathcal{O}_U = \mathcal{O}_X|_U)$ is surely a locally ringed space, as the stalks of $\mathcal{O}_X$ and $\mathcal{O}_U$ are the same. It suffices to prove that locally this pair is isomorphic to an affine scheme. To this end, we may assume that $X = \operatorname{Spec} A$. Since open sets of the form U_f, $f \in A$ form a base for the topology, and these open sets are affine, it suffices to observe that

$$(U_f, \mathcal{O}_X|_{U_f}) \simeq (U_f, \mathcal{O}_U|_{U_f}),$$

for any open set $U_f \subset U$ and to invoke (2.1).

2.3 (a) Suppose that the stalk $\mathcal{O}_{X,p}$ contains an element $f = (g, U)$ which is nilpotent. Since $f^n = 0$, possibly making U smaller, we may assume that $g^n = 0$, but $g \neq 0$. But then $\mathcal{O}_X(U)$ contains a nilpotent element.

Conversely, if $g \in \mathcal{O}_X(U)$ is nilpotent then pick a point $p \in U$ such that $g \neq 0 \in \mathcal{O}_{X,p}$. Then $f = (g, U) \in \mathcal{O}_{X,p}$ is also nilpotent.

(b) Clearly the pair $(X, \mathcal{O}_{X_{\text{red}}})$ is a locally ringed space, and so it suffices to prove that locally it is isomorphic to an affine scheme. To this end, we may assume that $X = \operatorname{Spec} A$ is affine. Note that there is a surjective ring homomorphism,

$$\phi \colon A \longrightarrow B,$$

where B is the quotient of A by the intersection of all the prime ideals, which is nothing but the set of all nilpotent elements of A. Since we have an equivalence of categories, this induces a morphism

$$(h, h^{\#}) \colon \operatorname{Spec} B \longrightarrow X = \operatorname{Spec} A.$$

This induces a morphism of sheaves, between the structure sheaf of $\operatorname{Spec} B$ and $\mathcal{O}_{X_{\text{red}}}$, which is an isomorphism, since it is an isomorphism stalk by stalk.

(c) It suffices to prove this locally on X and Y. Thus we may assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are both affine schemes. There is an induced ring homomorphism $\phi \colon A \longrightarrow B$. By assumption B contains no nilpotents, thus the kernel of ϕ contains the nilpotent elements $I \triangleleft A$, and there is a natural ring homomorphism $C \longrightarrow B$, where $C = A/I$. By (b) $Y_{\text{red}} = \operatorname{Spec} C$ and it is clear that the induced morphism $X \longrightarrow Y_{\text{red}}$ has the given universal property.

2.4 Let $V = \operatorname{Spec} A$. We first check that α is injective. Since, we can check this locally, we may assume that $X = \operatorname{Spec} B$ is affine, in which case we have already shown in lectures that α is bijective.

Now suppose that we are given a ring homomorphism $\phi\colon A \longrightarrow B = \Gamma(X, \mathcal{O}_X)$. Let $U_i = \operatorname{Spec} B_i$ be an open affine cover of X. Then there are natural ring homomorphisms $B \longrightarrow B_i$, given by restriction of sections. Composing we get ring homomorphisms $A \longrightarrow B_i$ and so get morphisms of schemes $U_i \longrightarrow V$. To show that we get a morphism $X \longrightarrow V$ it suffices to show that we get the same morphism on U_{ij}. But the two ways to get morphisms $U_{ij} \longrightarrow X$ both induce the same ring homomorphism $A \longrightarrow B_{ij}$, where $B_{ij} = \Gamma(U_{ij}, \mathcal{O}_X)$, since both homomorphisms are the composition of ϕ and the homomorphism $B \longrightarrow B_{ij}$ given by restriction. It follows that the two morphisms are the same, by what we already proved. Hence α is surjective.

2.5 The points of $\operatorname{Spec} \mathbb{Z}$ are the ideals generated by the prime numbers, which are closed points, together with the zero ideal, which is the generic point. The proper closed sets correspond to finite unions of prime numbers.

Since $\mathbb{Z}$ is an initial object in the category of rings $\operatorname{Spec} \mathbb{Z}$ is a terminal object in the category of affine schemes. Since every scheme is locally an affine scheme, it follows that $\operatorname{Spec} \mathbb{Z}$ is a terminal object in the category of schemes.

2.6 There are no prime ideals in R, so that the spectrum is empty. R is a terminal object for the category of rings, so it is an initial object for the category of affine schemes. But then it is an initial object for the category of schemes.

2.7 Since K is a field, it has a unique prime ideal, and so $\operatorname{Spec} K$ certainly has only one point, and the structure sheaf is represented by K itself. To give a morphism of $\operatorname{Spec} K$ to X, we certainly have to pick out a point $x \in X$. But then, by definition of a scheme, there is an induced morphism of local rings,

$$\mathcal{O}_{X,x} \longrightarrow K$$

But this is equivalent to a ring homomorphism, which sends the maximal ideal m_x to a point, which in turn is equivalent to giving an inclusion of the residue field of x into K.