MODEL ANSWERS TO THE FIRST HOMEWORK

1. If G is a group, then let $\mathcal{C}$ be the category with one object $*$, such that $\operatorname{Hom}(*, *) = G$, with composition of morphisms given by the group law in G. Then the identity in G plays the role of the identity morphism, the associative law in G gives associativity of composition, and the existence of inverses in G makes every morphism an isomorphism, and conversely.
2. Suppose that U maps to both W and Z over Y. Then U maps to both W and Z over X, as f is a monomorphism. But then there is a unique morphism to $W \underset{X}{\times} Z$ by the universal property of the fibre product. But then $W \underset{X}{\times} Z$ satisfies the universal property of the fibre product over Y and we are done by uniqueness of the fibre product.
3. (i) Clear.
(ii) The equaliser of f_1 and $f_2 \colon X \longrightarrow Y$ is the set

$$\{\, x \in X \mid f_1(x) = f_2(x) \,\},$$

together with its natural inclusion into X.
(iii) This is not correct. Let $\mathcal{C}$ be the category with one object and two morphisms. Then nothing equalises the two morphisms, but fibre products do exist. Indeed if the two morphisms are f and $g \colon X \dashrightarrow X$, then given any half of square, one can always fill it in with X at the top, selecting f and g as appropriate to go on the two edges.
In fact one needs to assume the existence of products in $\mathcal{C}$ as well:

Lemma 0.1. *Let $\mathcal{C}$ be a category which admits products.*
Then $\mathcal{C}$ admits equalisers iff it admits fibre products.

Proof. Suppose that $\mathcal{C}$ admits equalisers. Let $f \colon X \longrightarrow B$ and $g \colon Y \longrightarrow B$ be two morphisms. Then there are two morphisms $p \colon X \times Y \longrightarrow B$ (respectively q), the composition of projection down to X (respectively Y) and then f (respectively g) as appropriate. Let E be the equaliser of p and q. Then E maps to $X \times Y$, whence it maps to X and Y, via either projection, and these two morphisms become equal when composed with f and g. Now suppose that Z maps to both X and Y over B. Then it maps to $X \times Y$, and composing with projection down to X or Y and then f or g as appropriate. It follows that Z maps to E, by the universal property of the equaliser. But then E satisfies the universal property of the fibre product.

Now suppose that $\mathcal{C}$ admits fibre products. If f and $g\colon X \longrightarrow Y$ are two morphisms, then we get a morphism $X \longrightarrow Y \times Y$, by definition of the product. Note that there is also a morphism $\delta\colon Y \longrightarrow Y \times Y$ induced by the identity on both factors. Let $E = X \underset{X\times Y}{\times} Y$. Then E maps to X and composing this map with either f or g is the same. Suppose that Z maps to X, such that the composition with f or g is the same. Then Z maps to X and its maps to Y over $Y \times Y$. So Z maps to E, by the universal property of the fibre product. But then E satisfies the universal property of the equaliser. $\square$

4. (i) Sketched in the lecture notes.
(ii) Let I be an object of $\mathbb{I}$. By definition of α, we are given a morphism $\alpha(I)\colon F(I) \longrightarrow G(I)$. Since $\varinjlim_{\mathbb{I}} G$ is a prelimit, there are morphisms $G(I) \longrightarrow \varinjlim_{\mathbb{I}} G$. Composing, it follows that there are morphisms $F(I) \longrightarrow \varinjlim_{\mathbb{I}} G$. One can check easily that the construction of these morphisms is functorial with respect to morphisms $I \longrightarrow J$ in $\mathbb{I}$. By the universal property of the limit $\varinjlim_{\mathbb{I}} F$, there is then a morphism

$$\varinjlim_{\mathbb{I}}(\alpha)\colon \varinjlim_{\mathbb{I}} F \longrightarrow \varinjlim_{\mathbb{I}} G.$$

The rest is tedious checking.
(iii) Suppose we are given morphisms from T to W and X over Z. By composition, we are then given morphisms from T to W and Y over Z. By the universal property of the fibre product, there is then a unique morphism $T \longrightarrow Y \underset{Z}{\times} W$. So now we have morphisms from T to X and $Y \underset{Z}{\times} W$ over Y. By the universal property of the fibre product, there is then a unique morphism $T \longrightarrow X \underset{Y}{\times} (Y \underset{Z}{\times} W)$. So $X \underset{Y}{\times} (Y \underset{Z}{\times} W)$ satisfies the universal property of the fibre product.
Alternatively one can proceed as follows. Let $\mathbb{I}$ be the category defining the fibre product. Let F be the functor associated to W, Y and Z and let G be the functor associated to W, X and Z. Then there is a natural transformation $\alpha\colon G \longrightarrow F$. It associates to the three objects of $\mathbb{I}$ the maps $W \longrightarrow W$, $X \longrightarrow Y$ and $Z \longrightarrow Z$. The isomorphism we are looking for is then given by

$$\varprojlim_{\mathbb{I}}(\alpha)\colon \varprojlim_{\mathbb{I}} F \longrightarrow \varprojlim_{\mathbb{I}} G.$$

5. Suppose that we are given a natural transformation $u\colon h_Y \longrightarrow h_{Y'}$. Then we get a function $u_Y\colon h_Y(Y) \longrightarrow h_{Y'}(Y)$. Let $\phi = u_Y(i_Y) \in h_{Y'}$. Then $\phi\colon Y \longrightarrow Y'$ is a morphism.
It suffices to show that $u = h(\phi)$. If X is an object of $\mathcal{C}$ then we get a function $u_X\colon h_Y(X) \longrightarrow h_{Y'}(X)$. Pick $f \in h_Y(X)$. Since u is a natural transformation we get a commutative square

$$\begin{tikzcd}
h_Y(Y) \arrow[r, "u_Y"] \arrow[d, "h_Y(f)"'] & h_{Y'}(Y) \arrow[d, "h_{Y'}(f)"] \\
h_Y(X) \arrow[r, "u_X"] & h_{Y'}(X).
\end{tikzcd}$$

Consider starting at the top left hand corner, with the morphism i_Y. Going to the right and then down, we get $f \circ \phi$. Going down and then to the right, we get $u_X(f)$. Thus $u_X(f) = f \circ \phi = h(\phi)$. Since f and X were arbitrary, the result follows.