## 7. Rational, unirational and rationally connected varieties

In this section we give a geometric application of some of the ideas of the previous sections. Recall the definition of a rational variety.

**Definition 7.1.** *A variety $X$ over* $\operatorname{Spec} k$ *is* ***rational*** *if it birational to $\mathbb{P}^n_k$, for some $n$.*

**Proposition 7.2.** *Let $X$ be a variety over* $\operatorname{Spec} k$*.*

*Then the following are equivalent:*

1. *$X$ is rational.*
2. *An open subset of $X$ is isomorphic to an open subset of $\mathbb{P}^n_k$.*
3. *The function field of $X$ is a purely transcendental extension of $k$.*

It turns out that the question of which varieties are rational is one of the subtlest geometric problems one can ask. Note already one subtlety of this question, which does not arise in the classical case. Now that we are working with schemes, we are free to work over a non-algebraically closed field. It turns out that in this case this question can become very tricky, even in low dimensions.

**Definition 7.3.** *A* ***$k$-rational point*** *of a scheme $X$ oover $S$ is any point which is the image of a morphism* $\operatorname{Spec} k \longrightarrow X$ *over $S$. The set of all $k$-rational points is denoted $X(k)$.*

In other words a $k$-rational point is simply a point whose residue field is a subfield of $k$.

**Example 7.4.** *Let $X = \mathbb{A}^1_{\mathbb{R}}$. Then $p = \langle x^2 + a \rangle \in X$, where $a \in \mathbb{R}$ and $a > 0$, corresponds to two $\mathbb{C}$-valued points. Indeed, there are two scheme maps*

$$\operatorname{Spec} \mathbb{C} \longrightarrow X,$$

*whose image is $p$, corresponding to the fact that there are two automorphisms of* $\operatorname{Spec} \mathbb{C}$ *over* $\operatorname{Spec} \mathbb{R}$*, given by the identity and complex conjugation.*

**Lemma 7.5.** *Let $C \subset \mathbb{P}^2_k$ be a smooth conic, over a field $k$.*

*Then $C \simeq \mathbb{P}^1_k$ iff $C$ contains a $k$-rational point*

*Proof.* One direction is clear as $\mathbb{P}^1_k$ certainly contains $k$-rational points. Now suppose that $C$ contains a $k$-rational point. After applying an element of $\mathrm{PGL}(3, k)$, we may assume that this point is $[0 : 0 : 1]$. Consider projection from this point. This defines a morphism $C -$

$[0:0:1] \longrightarrow \mathbb{P}^1$, which is surely defined over $k$ (indeed it is the restriction of $[x:y:z] \longrightarrow [x:y]$). It is then straightforward to check that this morphism extends to an isomorphism. □

**Example 7.6.** *The conic* $C = V(x^2 + y^2 - z^2) \subset \mathbb{P}^2_{\mathbb{R}}$ *is not rational over* $\operatorname{Spec} \mathbb{R}$.

Since the problem of determining whether a variety is rational or not is so delicate, various intermediary notions have been introduced.

**Definition 7.7.** *We say that a variety* $X$ *is* ***unirational*** *if there is a dominant rational map* $\mathbb{P}^n_k \dashrightarrow X$.

Note some basic properties of unirational varieties.

**Lemma 7.8.** *Let* $X$ *be a variety over* $k$*. The following are equivalent:*

(1) $X$ *is unirational.*
(2) *There is a dominant generically finite morphism* $\phi\colon Y \longrightarrow X$*, where* $Y$ *is rational.*
(3) *The function field of* $X$ *is contained in a purely transcendental field extension of* $k$.
(4) *There is a finite extension of the function field of* $X$ *which is a purely transcendental field extension of* $k$.

*Proof.* The fact that (1) and (3) are equivalent, follows from the equivalence of categories between dominant rational maps and inclusions of function fields, and (4) follows from (2) in a similar fashion.

So suppose that $\phi\colon \mathbb{P}^n_k \dashrightarrow X$ is a dominant rational map. Replacing $\mathbb{P}^n$ by the normalisation of the graph of $\phi$, we may assume that there a quasi-projective variety $Y$ and a dominant morphism $Y \longrightarrow X$. If the dimension of the generic fibre is greater than zero, then pick a hyperplane $H \subset \mathbb{P}^n$, whose inverse image in $Y$ dominates $X$. Continuing in this way, we reduce to the case where is generically finite. □

**Question 7.9** (Lüroth's problem)**.** *Is every unirational variety rational?*

Note that one way to restate Lüroth's problem is to ask if every subfield of a purely transcendental field extension is purely transcendental. It turns out that the answer is yes in dimension one, in all characteristics. This is typically a homework problem in a course on Galois Theory. There is also a simple geometric proof of this fact (essentially the Riemann-Hurwitz formula). In dimension two the problem is already considerably harder, and it is false if one allows inseparable field extensions.

In dimension three it was shown to be false even in characteristic zero, in 1972, using three different methods.

One proof is due to Artin and Mumford. It had been observed by Serre that the cohomology ring of a smooth unirational threefold is indistinguishable from that of a rational variety (for $\mathbb{P}^3$ one gets $\mathbb{Z}[x]/\langle x^3 \rangle$ and the cohomology ring varies in a very predictable under blowing up and down) except possibly that there might be torsion in $H_3(X, \mathbb{Z})$. They then give an reasonably elementary construction of a threefold with non-zero torsion in $H_3$.

Another proof is due to Clemens and Griffiths. It is not hard to prove that every smooth cubic hypersurface in $\mathbb{P}^4$ is unirational. On the other hand they prove that some smooth cubics are not rational. To prove this consider the family of lines on the cubic. It turns out that this is a two dimensional family, and that a lot of the geometry of the cubic is controlled by the geometry of this surface.

The third proof is due to Iskovskikh and Manin. They prove that every smooth quartic in $\mathbb{P}^4$ is not rational. On the other hand, some quartics are unirational. In fact they show, in an amazing tour de force, that the birational automorphism group of a smooth quartic is finite. Clearly this means that a smooth quartic is never rational.

Let us see how to prove that a smooth cubic threefold is unirational.

**Definition 7.10.** *Let $\pi \colon X \longrightarrow S$ be a morphism of schemes. A* ***section*** *of is a morphism $\sigma \colon S \longrightarrow X$ such that $\sigma \circ \pi$ is the identity. A rational section is a section defined on some open subset $U$ of $S$.*

**Lemma 7.11.** *Let $\pi \colon X \longrightarrow S$ be a morphism of integral schemes, of finite type. Then picking a rational section of $\pi$ is equivalent to picking a rational point of the generic fibre.*

*Proof.* Let $K$ be the function field of $S$. We may as well assume that both $S = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$ are affine, so that $K$ is the field of fractions of $A$. The generic fibre has coordinate ring $B \underset{A}{\otimes} K$. Suppose that we have a rational section. Then we may as well assume that we have a section. But this is equivalent to a ring homomorphism $B \longrightarrow A$. In turn this induces a ring homomorphism $B \underset{A}{\otimes} K \longrightarrow K$ which is equivalent to a morphism $\operatorname{Spec} K \longrightarrow X_\xi$, where $\xi$ is the generic point of $S$. But this is exactly the same as a rational point of the general fibre.

Now suppose that we have a rational point of the generic fibre. This is equivalent to a ring homorphism $B \underset{A}{\otimes} K \longrightarrow K$. Since we have a morphism of finite type, $B$ is a finitely generated $A$-algebra. Pick generators $b_1, b_2, \dots, b_k$. Denote the image of $b_i$ in $K$ by $c_i/d_i$, where $c_i$ and $d_i$ are elements of $A$. Passing to the open affine subset $U_d$ of $S$, where $d$ is the product $d_1 \cdot d_2 \cdot \dots \cdot d_k$, we may assume that $d_i = 1$, so that we get a morphism $B \longrightarrow A$. But this is equivalent to a section of $\pi$. $\square$

**Definition 7.12.** *Let $S$ be a scheme.* ***Projective n-space over** $S$, denoted $\mathbb{P}^n_S$, is the scheme obtained by base change of $\mathbb{P}^n_{\mathbb{Z}}$, where $S \longrightarrow$* Spec $\mathbb{Z}$ *is the canonical morphism. A* ***projective morphism*** *over $S$, $\pi\colon X \longrightarrow S$ is any morphism, which admits a factorisation into a closed immersion $X \longrightarrow \mathbb{P}^n_S$ over $S$.*

*A* ***conic bundle*** *is a projective morphism, where $n = 2$ and the fibres are conics. A rational conic bundle, is any morphism, which is a conic bundle over an open subset of the base.*

Of course the fibres of any conic bundle have three types

- a smooth conic,
- a pair of lines,
- a double line.

In the case when $X$ and $S$ are integral schemes, a morphism is a rational conic bundle iff the generic fibre is a smooth conic in $\mathbb{P}^2_K$, where $K$ is the function field of the base.

**Proposition 7.13.** *Let $\pi\colon X \longrightarrow S$ be a rational conic bundle, between two varieties, over an algebraically closed field $k$. Let $T \subset X$ be a subvarety of $X$ which dominates $S$.*

(1) *If $T$ is unirational, then so is $X$.*
(2) *If $T \longrightarrow S$ is birational and $T$ is rational, then so is $X$.*

*Proof.* Consider the base change $T \longrightarrow S$ of $X$. Let $Y$ be a component of the base change of maximal dimension, which dominates $X$. Then $Y \longrightarrow T$ is a conic bundle. Moreover, there is a natural morphism $T \longrightarrow Y$ which is a section. Possibly base changing further, we may assume that the base is rational, and that there is a rational section. Thus it suffices to prove (2).

Consider the generic fibre. By assumption it is a smooth conic in $\mathbb{P}^2_K$, where K is not algebraically closed. The rational section implies that this conic has a rational point. But then this conic is isomorphic to $\mathbb{P}^1_K$. The function field of this conic is then $K(t)$. The generic point of $X$ is also the ceneric point of the generic fibre. It follows that the function field of $X$ is isomorphic to $K(t)$. Since $K$ is purely transcendental over $k$ the groundfield, this implies that the field of fractions of $X$ is purely transendental over $k$. But then $X$ is rational. $\square$

To finish off, we have to exhibit a conic bundle structure on a smooth cubic threefold.

**Lemma 7.14.** *Let $S$ be a smooth cubic surface in $\mathbb{P}^3$.*

*Then $S$ contains twenty seven lines.*

*Proof.* A cubic is specified by choosing the coefficients of a homogeneous cubic in four variables of which there are $\binom{6}{3} = 20$; the space of all cubics is therefore naturally parametrised by $\mathbb{P}^{19}$. Consider the incidence correspondence

$$\Sigma = \{\, (l, F) \in \mathbb{G}(1,3) \times \mathbb{P}^{19} \,|\, l \subset V(F) \,\} \subset \mathbb{G}(1,3) \times \mathbb{P}^{19}.$$

Then this is a closed subset of $\mathbb{G}(1,3) \times \mathbb{P}^{19}$ and the two natural projections $f \colon \Sigma \longrightarrow \mathbb{G}(1,3)$ and $g \colon \Sigma \longrightarrow \mathbb{P}^{19}$ are proper (the image of a closed set is closed; in fact $f$ and $g$ are projective).

Let $p \in \mathbb{G}(1,3)$ and consider $f^{-1}(l)$. This is the space of cubics containing the line $l$. There are two ways to figure out what the fibre looks like. Firstly one can change coordinates so that $l = V(X_2, X_3)$, so that the points of $l$ are $[a : b : 0 : 0]$. In this case the coefficients of $X^3$, $X^2Y$, $XY^2$ and $Y^3$ must all vanish. The fibre is a copy of a linear subspace of dimension 15 in $\mathbb{P}^{19}$. Alternatively pick four distinct points $p_1$, $p_2$, $p_3$ and $p_4$ of $l$. The condition that $F(p_i) = 0$ imposes one linear constraint. One can check that these four points impose independent conditions, so that that the space of cubics containing all four points is a linear subspace of dimension 15. Suppose $F(p_i) = 0$, for $1 \leq i \leq 4$. Then $F|_l$ is a cubic polynomial in two variables, vanishing at four points. Thus $F|_l$ is the zero polynomial, $l \subset V(F)$ iff $F$ vanishes at $p_i$, for $1 \leq i \leq 4$. Thus $\Sigma$ fibres over an irreducible base with irreducible fibres of the same dimension. It follows that $\Sigma$ is irreducible of dimension $4 + 15 = 19$.

It suffices then to exhibit a single cubic with twenty seven lines, since then the morphism $g$ must be dominant, whence surjective. It is a fun exercise to compute the 27 lines on $X^3 + Y^3 + Z^3 + T^3 = 0$. □

**Lemma 7.15.** *Let $V$ be a smooth cubic threefold in $\mathbb{P}^4$. Then $V$ contains a*

(1) *two dimensional family of lines, and a*
(2) *four dimensional family of conics.*

*Proof.* Let $H$ be a general hyperplane in $\mathbb{P}^4$. Then $S = H \cap V$ is a smooth cubic surface in $\mathbb{P}^3$. But then we have already seen that $S$ contains a finite number of lines. Given any line $l$ in $S$, then any hyperplane containing $l$ cuts out $l$ union a residual conic. Conversely given any conic in $S$, the hyperplane spanned by the conic, cuts out a residual line. Thus the family of conics contained in $S$ has dimension one.

But if one fixes a line $l$ then there is a two dimensional family of hyperplanes that contains it, and if one fixes a conic there is a one dimensional family of hyperplanes containing the conic. Since there is a four dimensional family of hyperplanes, it follows that there is a two dimensional family of lines on $V$ and a four dimensional family of conics. $\square$

Another way to restate (7.15) is that there are a finite number of lines through the general point of $V$, but a two dimensional family of conics. Let $V$ be a smooth cubic in $\mathbb{P}^4$. Let $l$ be a line contained in $V$ and consider the family of planes which contains $l$. This family is isomorphic to $\mathbb{P}^2$. On the other hand, any plane intersects $V$ in a plane cubic. Thus a plane which contains $l$ intersects $V$ in a residual plane conic. The family of all planes that contain $V$ defines a conic bundle on $W$, the total space of the family of residual conics. Since there is a unique plane containing $l$ through any point of $V$ away from $l$, $W$ is birational to $V$ and is obtained from $V$ by blowing up the locus $l$. Let $E$ be the corresponding exceptional divisor. The fibres of the morphism $E \longrightarrow l$ are lines in the corresponding planes. Thus $E$ is rational and the morphism $E \longrightarrow S$, where $S$ is the base of the family of conics, is two to one, since a general line will meet a conic in two points. It follows by (7.13) that $V$ is indeed unirational.

Since it is so hard to distinguish between rational and unirational, yet another closely related notion has been introduced.

**Definition 7.16.** *Let $X$ be a variety, over an algebraically closed field of characteristic zero. We say that $X$ is **rationally connected** if for two general points $x$ and $y$ of $X$, we may find a rational curve connecting $x$ and $y$.*

One convenient way to restate this condition, is that for two general points $x$ and $y$, we may find a morphism

$$f\colon \mathbb{P}^1_k \longrightarrow X,$$

such that $f(0) = x$ and $f(\infty) = y$. Indeed the nonconstant image of $\mathbb{P}^1_k$ is always birational to $\mathbb{P}^1_k$.

Note the following basic properties of rationally connected varieties. If $X$ is rationally connected and $X \dashrightarrow Y$ is a dominant rational map, then $Y$ is rationally connected. On the other hand $\mathbb{P}^n_k$ is rationally connected; indeed just take the line connecting the two points. It follows that every unirational variety is rationally connected. It is expected that the converse is false. In fact, if $X \subset \mathbb{P}^n_k$ is a hypersurface of degree $d$, then X is rationally connected iff $d \leq n$. On the other hand, it is expected that the general quartic in $\mathbb{P}^4$ is not rationally connected. However to prove this seems one of the most challenging problems in birational geometry.

Rationally connected varieties have some other very nice properties.

**Definition 7.17.** *Let $X$ be a smooth variety, over an algebraically closed field of characteristic zero. We say that $X$ is* ***rationally chain connected*** *if for two general points $x$ and $y$ of $X$, we may find a cchain of rational curves connecting $x$ and $y$,*

$$C = \bigcup_{i=0}^{k} C_i,$$

*where $x \in C_0$, $y \in C_k$ and $C_i \cap C_{i+1} \neq \emptyset$.*

**Theorem 7.18.** *Let $X$ be a smooth variety over an algebraically closed field of characteristic zero.*

*The following are equivalent*

1. *For every two points $x$ and $y \in X$ we may find a raitonal curve connecting $x$ to $y$.*
2. *$X$ is rationally connected,*
3. *$X$ is rationally chain connected.*

**Theorem 7.19** (Deformation Invariance)**.** *Let $\pi \colon X \longrightarrow B$ be a family of smooth varieties, over an algebraically closed field of characteristic zero. Suppose that $b$ and $b'$ belong to the same connected component of $B$.*

*Then $X_b$ is rationally connected iff $X_{b'}$ is rationally connected.*

Just recently, the following deep result has been proved concerning rationally connected varieties.

**Theorem 7.20.** *Let $\pi \colon X \longrightarrow B$ be a morphism of varieties, over an algebraically closed field of characteristic zero.*

*If both the generic fibre of $\pi$ and $B$ are rationally connected then $X$ is rationally connected.*

We sketch some of the details of the proof. The first key fact, proved by Kollár, Miyaoka and Mori, is that given any variety over an algebraically closed field of characteristic zero, there is a maximal rationally connected fibration.

**Theorem 7.21.** *Let $X$ be a variety over an algebraically closed field of characteristic zero. Then there is a fibration $\pi \colon X \dashrightarrow B$, whose generic fibre is rationally connected, such that any other such rational map fibre through $\pi$.*

We now need a few simple results.

**Definition 7.22.** *A variety $X$ is said to be* ***ruled*** *if it is birational to $Y \times \mathbb{P}^1$. We say that $X$ is* ***uniruled*** *if $X$ is dominated by a ruled variety.*

**Lemma 7.23.** *A variety $X$ is uniruled iff for every general point $x$ we can find a rational curve $C$ containing $x$.*

**Lemma 7.24.** *To prove* (7.20)*, it suffices to prove that if $\pi\colon X \longrightarrow \mathbb{P}^1$ is a morphism of varieties over an algebraically closed field of characteristic zero, such that the generic fibre of $\pi$ is rationally connected, then $\pi$ has a section.*

*Proof.* Let $\pi\colon X \longrightarrow B$ be a morphism of varieties, over an algebraically closed field of characteristic zero, where both the generic fibre of $\pi$ and $B$ are rationally connected. By (7.18), it suffices to prove that $X$ is rationally chain connected.

Let $x$ and $y \in X$. Let $b$ be the image of $x$ in $B$, $c$ the image of $y$. Since $B$ is rationally connected, we may find a chain of rational curves $C$ in $B$ connecting $b$ to $c$. By induction on the length of the chain, we may assume that the chain connecting $b$ to $c$ has length one. Pulling back to $\mathbb{P}^1$, we may as well assume that the base is $\mathbb{P}^1$. Assume that we can find a section $\sigma$. Let $\Sigma = \sigma(B)$, so that $\Sigma$ is a rational curve dominating $B$. Let $x' = \sigma(b)$, and $y' = \sigma(c)$.

As the generic fibre is rationally connected both of the fibres $\pi^{-1}(b)$ and $\pi^{-1}(y)$ are rationally connected. Pick a rational curve $C_b$ connecting $x$ to $x'$ and a rational curve $C_c$ connecting $y$ to $y'$, whose existence is guaranteed by (7.18). Then $C = C_b \cup \Sigma \cup C_c$ is a rational chain connecting $x$ to $y$. But then $X$ is rationally connected by (7.18). ∎

Now to prove that there is always a section, the key point is to establish that every fibre is reduced. Indeed, a section intersects the general fibre in one reduced point, so that the intersection number is one. Therefore the intersection number with every fibre is one. But if $F = mG$, (that is $F$ is a fibre with multiplicity $m$) then

$$\Sigma \cdot F = m(\Sigma \cdot G) \geq m.$$

Thus $m = 1$.