5. Schemes

To define schemes, just as with algebraic varieties, the idea is to first define what an affine scheme is, and then realise an arbitrary scheme, as something which is locally an affine scheme. The definition of an affine scheme is motivated by the correspondence between affine varieties and finitely generated algebras over a field, without nilpotents. The idea is that we should be able to associate to any ring R, a topological space X, and a set of continuous functions on X, which is equal to R. In practice this is too much to expect and we need to work with a slightly more general object than a continuous function.

Now if X is an affine variety, the points of X are in correspondence with the maximal ideals of the coordinate ring $A = A(X)$. Unfortunately if we have two arbitrary rings R and S, then the inverse image of a maximal ideal won't be maximal. However it is easy to see that the inverse image of a prime ideal is a prime ideal.

Definition 5.1. *Let R be a ring. $X = \operatorname{Spec} R$ denotes the set of prime ideals of R. X is called the* ***spectrum*** *of R.*

Note that given an element of R, we may think of it as a function on X, by considering it value in the quotient.

Example 5.2. *It is interesting to see what these functions look like in specific cases. Suppose that we take $X = \operatorname{Spec} k[x, y]$. Now any element $f = f(x, y) \in k[x, y]$ defines a section of $\mathcal{O}_X(X)$. Suppose that we consider a maximal ideal of the form $\mathfrak{p} = \langle x - a, x - b\rangle$. Then the value of f at $\mathfrak{p}$ is equal to the class of f inside the quotient*

$$R/\mathfrak{p} = \frac{k[x, y]}{\langle x - a, x - b\rangle}.$$

If we identify the quotient with k, under the obvious identification, then this is the same as evaluating f at (a, b). Now consider $\mathbb{Z}$. Suppose that we choose an element $n \in \mathbb{Z}$. Then the value of n at the prime ideal $\mathfrak{p} = \langle p\rangle$ is equal to the value of n modulo p. For example, consider $n = 60$. Then the value of this function at the point 7 is equal to $60 \mod 7 = 4 \mod 7$. Moroever 60 has zeroes at 2, 3 and 5, where both 3 and 5 are ordinary zeroes, but 2 is a double zero.

Suppose that we take the ring $R = \mathbb{Z}[x]/\langle x^2\rangle$. Then the spectrum contains only one element, the prime ideal $\langle x\rangle$. Consider the element $x \in R$. Then x is zero on the unique element of the spectrum, but it is not the zero element of the ring.

Now we wish to define a topology on the spectrum of a ring. We want to make the functions above continuous. So given an element

$f \in R$, we want the set

$$\{\, \mathfrak{p} \mid f(\mathfrak{p}) = 0 \,\} = \{\, \mathfrak{p} \mid \langle f \rangle \subset \mathfrak{p} \,\},$$

to be closed. Given that any ideal $\mathfrak{a}$ is the union of all the principal ideals contained in it, so that the set of prime ideals which contain $\mathfrak{a}$ is equal to the intersection of prime ideals which contain every principal ideal contained in $\mathfrak{a}$ and given that the intersection of closed sets is closed, we have an obvious candidate for the closed sets:

Definition 5.3. *The* ***Zariski topology*** *on X is given by taking the closed sets to be*

$$V(\mathfrak{a}) = \{\mathfrak{p} \in R \mid \mathfrak{a} \subset \mathfrak{p}\,\},$$

where $\mathfrak{a}$ is any ideal of R.

Lemma 5.4. *Let R be a ring. Then $X = \operatorname{Spec} R$ is a topological space Moreover the open sets*

$$U_f = \{\, \mathfrak{p} \in R \mid f \notin \mathfrak{p} \,\},$$

form a base for the topology.

Proof. Easy check. □

By what we said above, the Zariski topology is the weakest topology so that the zero sets of $f \in R$ are closed.

Example 5.5. *Let k be a field. Then* Spec k *consists of a single point. Now consider* Spec $k[x]$*. If k is an algebraically closed field, then by the Nullstellensatz, the maximal ideals are in correspondence with the points of k. However, since $k[x]$ is an integral domain, the zero ideal is a prime ideal. Since $k[x]$ is a PID, the proper closed sets of X consist of finite unions of maximal ideals. The closure of the point $\xi = \langle 0 \rangle$ is then the whole of X. In particular, not only is the Zariski topology, for schemes, not Hausdorff or T_2, it is not even T_1. Now consider $k[x, y]$, where k is an algebraically closed field. Prime ideals come in three types. The maximal ideals correspond to points of k^2. The zero ideal, whose closure consists of the whole of X. But there are also the prime ideals which correspond to prime elements $f \in k[x, y]$. The zero locus of f is then an irreducible curve C, and in fact the closure of the point $\xi = \langle f \rangle$ is then the curve C. The proper closed sets thus consist of a finite union of maximal ideals, union infinite sets of maximal ideal which consist of all points belonging to an affine curve C, together with the ideal of each such curve.*

Now suppose that k is not algebraically closed. For example, consider Spec $\mathbb{R}[x]$*. As before the closure of the zero ideal consists of the whole of X. The maximal ideals come in two flavours. First there are the* *ideals $\langle x-a\rangle$, where $a \in \mathbb{R}$. But in addition there are also the ideals x^2+a, where a is a positive real number. The set $V(x^2+y^2=-1)$ does not contain any ideals of the first kind, but it contains many ideals of the second kind.*

Now suppose that we take $\mathbb{Z}$. In this case the maximal ideals correspond to the prime numbers, and in addition there is one point whose closure is the whole spectrum. In this respect $\operatorname{Spec}\mathbb{Z}$ *is very similar to* $\operatorname{Spec} k[t]$.

Lemma 5.6. *Let X be the spectrum of the ring R and let $f \in R$.*

If $U_f = \bigcup U_{g_i}$ then $f^n = \sum b_i g_i$, where $b_1, b_2, \dots, b_k \in R$. In particular U_f is compact.

Proof. Suppose that we are given an open cover of U_f. Since sets of the form U_g form a base for the Zariski topology, we may assume that

$$U_f = \bigcup_i U_{g_i}.$$

Taking complements, we see that

$$V(\langle f\rangle) = \bigcap_i V(\langle g_i\rangle) = V(\langle \sum_i g_i\rangle).$$

Now $V(\mathfrak{a})$ consists of all prime ideals that contain $\mathfrak{a}$, and the radical of $\mathfrak{a}$ is the intersection of all the prime ideals that contain $\mathfrak{a}$. Thus

$$\sqrt{\langle f\rangle} = \sqrt{\langle \sum_i g_i\rangle}.$$

But then, in particular, f^n is a finite linear combination of the g_i and the corresponding open sets cover U_f. □

As pointed out above, we need a slightly more general notion of a function than the one given above:

Definition 5.7. *Let R be a ring. We define a sheaf of rings $\mathcal{O}_X$ on the spectrum of R as follows. Let U be any open set of X. A section σ of $\mathcal{O}_X(U)$ is by definition any function*

$$s\colon U \longrightarrow \coprod_{\mathfrak{p}\in U} R_{\mathfrak{p}},$$

where $s(\mathfrak{p}) \in R_{\mathfrak{p}}$, which is locally represented by a quotient. More precisely, given a point $\mathfrak{q} \in U$, there is an element $f \in R$ such that $U_f \subset U$ and such that the section $\sigma|_U$ is represented by a/f^n, for some $a \in R$.

An ***affine scheme*** *is then any locally ringed space isomorphic to the spectrum of a ring with its associated sheaf. A* ***scheme*** *is a locally* *ringed space, which is locally isomorphic, as locally ringed space, to an affine scheme.*

It is not hard to see that $\mathcal{O}_X(U)$ is a ring (sums and products are defined in the obvious way) and that we do in fact have a sheaf rather than just a presheaf.

The key result is the following:

Lemma 5.8. *Let X be an affine scheme, isomorphic to the the spectrum of R and let $f \in R$.*

1. *For any $\mathfrak{p} \in X$, the stalk $\mathcal{O}_{X,\mathfrak{p}}$ is isomorphic to the local ring $R_{\mathfrak{p}}$.*
2. *The ring $\mathcal{O}_X(U_f)$ is isomorphic to R_f.*

In particular $\mathcal{O}_X(X) \simeq R$.

Proof. We first prove (1). There is an obvious ring homomorphism

$$\mathcal{O}_{X,\mathfrak{p}} \longrightarrow R_{\mathfrak{p}},$$

which just sends a germ (g, U) to its value $g(\mathfrak{p})$ at $\mathfrak{p}$.

On the other hand, there is an obvious ring homomorphism,

$$R \longrightarrow \mathcal{O}_{X,\mathfrak{p}},$$

which sends an element $r \in R$ to the pair (r, X). Suppose that $f \notin \mathfrak{p}$. Then $(1/f, U_f)$ defines an element of $\mathcal{O}_{X,\mathfrak{p}}$, and this element is an inverse of (f, X). It follows, by the universal property of the localisation, that there is a ring homomorphism,

$$R_{\mathfrak{p}} \longrightarrow \mathcal{O}_{X,\mathfrak{p}},$$

which is the inverse map. Hence (1).

Now we turn to the proof of (2). As before there is an obvious ring homomorphism,

$$R \longrightarrow \mathcal{O}_X(U_f),$$

which induces a ring homomorphism

$$R_f \longrightarrow \mathcal{O}_X(U_f).$$

We have to show that this map is an isomorphism. We first consider injectivity. Suppose that $a/f^n \in R_f$ is sent to zero. Then the image of a/f^n is equal to zero in $R_{\mathfrak{p}}$, for every $f \notin \mathfrak{p}$. But then there is an element $h \notin \mathfrak{p}$ such that $ha = 0$ in R. Let $\mathfrak{a}$ be the annihilator of a in R. Then $h \in \mathfrak{a}$ and $h \notin \mathfrak{p}$, so that $\mathfrak{a}$ is not a subset of $\mathfrak{p}$. Since this holds for every $\mathfrak{p} \in U_f$, it follows that $V(\mathfrak{a}) \cap U_f = \emptyset$. But then $f \in \sqrt{\mathfrak{a}}$ so that $f^l \in \mathfrak{a}$, for some l. It follows that $f^l a = 0$, so that a/f^n is zero in R_f.

Now consider surjectivity. Pick $s \in \mathcal{O}_X(U_f)$. By assumption, we may cover U_f by open sets V_i such that s is represented by a_i/g_i on V_i. By definition $g_i \notin \mathfrak{p}$, for every $\mathfrak{p} \in V_i$, so that $V_i \subset U_{g_i}$. Now since sets of the form U_h form a base for the topology, we may assume that $V_i = U_{h_i}$. As $U_{h_i} \subset U_{g_i}$ it follows that $V(g_i) \subset V(h_i)$ so that

$$\sqrt{\langle h_i \rangle} \subset \sqrt{\langle g_i \rangle}.$$

But then $h_i^{n_i} \in \langle g_i \rangle$, so that $h_i^{n_i} = c_i g_i$. In particular

$$\frac{a_i}{g_i} = \frac{c_i a_i}{h_i^{n_i}}.$$

Replacing h_i by $h_i^{n_i}$ and a_i by $c_i a_i$, we may assume that U_f is covered by U_{h_i}, and that s is represented by a_i/h_i on U_{h_i}.

Now observe that (5.6), $f^n = \sum b_i h_i$, where $b_1, b_2, \dots, b_k \in R$ and U_f can be covered by finitely many of the sets U_{h_i}. Thus we may assume that we have only finitely many h_i. Now on $U_{h_i h_j} = U_{h_i} \cap U_{h_j}$, there are two ways to represent s, one way by a_i/h_i and the other by a_j/h_j. By injectivity, we have $a_i/h_i = a_j/h_j$ in $R_{h_i h_j}$ so that for some n,

$$(h_i h_j)^n (h_j a_i - h_i a_j) = 0.$$

Since there are only finitely many i and j, we may assume that n is independent of i and j. We may rewrite this equation as

$$h_j^{n+1}(h_i^n a_i) - h_i^{n+1}(h_j^n a_j) = 0.$$

If we replace h_i by h_i^{n+1} and a_i by $h_i^n a_i$, then s is still represented by a_i/h_i and moreover

$$h_j a_i = h_i a_j.$$

Let $a = \sum_i b_i a_i$, where $f^n = \sum_i b_i h_i$. Then for each j,

$$\begin{aligned} h_j a &= \sum_i b_i a_i h_j \\ &= \sum_i b_i h_i a_j \\ &= f^n a_j. \end{aligned}$$

But then $a/f^n = a_j/h_j$ on U_{h_j}. But then a/f^n represents s on the whole of U_f. □

Note that by (2) of (5.8), we have achieved our aim of constructing a topological space from an arbitrary ring R, which realises R as a natural subspace of the continuous functions.

Definition 5.9. *A morphism of schemes is simply a morphism between two locally ringed spaces which are schemes.*

The gives us a category, the category of schemes. Note that the category of schemes contains the category of affine schemes as a full subcategory and that the category of schemes is a full subcategory of the category of locally ringed spaces.

Theorem 5.10. *There is an equivalence of categories between the category of affine schemes and the category of commutative rings with unity.*

Proof. Let F be the functor that associates to an affine scheme, the global sections of the structure sheaf. Given a morphism

$$(f, f^{\#})\colon (X, \mathcal{O}_X) = \operatorname{Spec} B \longrightarrow (Y, \mathcal{O}_Y) = \operatorname{Spec} A,$$

of locally ringed spaces then let

$$\phi\colon A \longrightarrow B$$

be the induced map on global sections. It is clear that F is then a contravariant functor and F is essentially surjective by (5.8).

Now suppose that $\phi\colon A \longrightarrow B$ is a ring homomorphism. We are going to construct a morphism

$$(f, f^{\#})\colon (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y),$$

of locally ringed spaces. Suppose that we are given $\mathfrak{p} \in X$. Then $\mathfrak{p}$ is a prime ideal of B. But then $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$ is a prime ideal of A. Thus we get a function $f : X \longrightarrow Y$. Now if $\mathfrak{a}$ is an ideal of A, then $f^{-1}(V(\mathfrak{a})) = V(\langle\phi(\mathfrak{a})\rangle)$, so that f is certainly continuous. For each prime ideal $\mathfrak{p}$ of B, there is an induced morphism

$$\phi_{\mathfrak{p}}\colon A_{\phi^{-1}(\mathfrak{p})} \longrightarrow B_{\mathfrak{p}},$$

of local rings. Now suppose that $V \subset Y$ is an open set. We want to define a ring homomorphism

$$f^{\#}(V)\colon \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}(V)).$$

Suppose first that $V = U_g$, where $g \in A$. Then $\mathcal{O}_Y(V) = A_g$ and $f^{-1}(V) \subset U_{\phi(g)}$. But then there is a restriction map

$$\mathcal{O}_X(U_{\phi(g)}) \simeq B_{\phi(g)} \longrightarrow \mathcal{O}_X(f^{-1}(V)).$$

On the other hand, composing there is a ring homomorphism

$$A \longrightarrow B_{\phi(g)}.$$

Since the image of g is invertible, by the universal property of the localisation, there is an induced ring homomorphism

$$A_g \longrightarrow B_{\phi(g)}.$$

Putting all of this together, we have defined $f^{\#}(V)$ when $V = U_g$. Since the sets U_g form a base for the topology, and these maps are compatible in the obvious sense, this defines a morphism

$$f^{\#} \colon \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X,$$

of sheaves. Clearly the induced map on local rings is given by $\phi_{\mathfrak{p}}$, and so $(f, f^{\#})$ is a morphism of local rings.

Finally it suffices to prove that these two assignments are inverse. The composition one way is clear. If we start with ϕ and construct $(f, f^{\#})$ then we get back ϕ on global sections. Conversely suppose that we start with $(f, f^{\#})$, and let ϕ be the map on global sections. Given $\mathfrak{p} \in X$, we get a morphism of local rings on stalks, which is compatible with ϕ and localisation, so that we get a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A_{f(\mathfrak{p})} & \xrightarrow{f^{\#}_{\mathfrak{p}}} & B_{\mathfrak{p}}. \end{array}$$

But since $f^{\#}_{\mathfrak{p}}$ is a morphism of local rings, it follows that $\phi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$, so that f coincides with the map induced by ϕ. But then $f^{\#}$ is also the map induced by ϕ. ☐

Definition 5.11. *Let X be a scheme and let $x \in X$ be a point of X. The* ***residue field of** X **at*** *x is the quotient of $\mathcal{O}_{X,x}$ by its maximal ideal.*

Example 5.12. *Let R be the local ring of a curve over an algebraically closed field (or more generally a discrete valuation ring). Then* Spec R *consists of two points; the maximal ideal, and the zero ideal. The first t_0 is closed and has residue field the groundfield k of C, the second t_0 has residue field the quotient ring K of R, and its closure is the whole of X. The inclusion map $R \longrightarrow K$ corresponds to a morphism which sends the unique point of* Spec K *to t_1.*

There is another morphism of ringed spaces which sends the unique point of Spec K *to t_0 and uses the inclusion above to define the map on structure sheaves. Since there is only one way to map R to K, this does not come from a map on rings. In fact the second map is not* *a morphism of locally ringed spaces, and so it is not a morphism of schemes.*

Example 5.13. *It is interesting to see an example, in a seemingly esoteric case. Consider the case of a number field k (that is a finite extension of $\mathbb{Q}$, with its ring of integers $A \subset k$ (that is the integral closure of $\mathbb{Z}$ inside k). As a particular example, take $k = \mathbb{Q}(\sqrt{3})$. Then $A = \mathbb{Z} \oplus \mathbb{Z}\langle\sqrt{3}\rangle$. The picture is very similar, as for the case of $\mathbb{Z}$. There are infinitely many maximal ideals, and only one point which is not closed, the zero ideal. Moreover, as there is a natural ring homomorphism $\mathbb{Z} \longrightarrow A$, by our equivalence of categories, there is an induced morphism of schemes $\operatorname{Spec} A \longrightarrow \operatorname{Spec} \mathbb{Z}$. We investigate this map. Consider the fibre over a point $\langle p\rangle \in \operatorname{Spec} \mathbb{Z}$. This is just the set of primes in A containing the ideal pA. It is well known by number theorists, that three things can happen:*

(1) *If p divides the discriminant of $k/\mathbb{Q}$ (which in this case is 12), that is $p = 2$ or 3, then the ideal $\langle p\rangle$ is a square in A.*

$$\langle 2\rangle A = \langle -1 + \sqrt{3}\rangle^2,$$

and

$$\langle 3\rangle A = \langle\sqrt{3}\rangle^2.$$

(2) *If 3 is a square modulo p, the prime $\langle p\rangle$ factors into a product of distinct primes,*

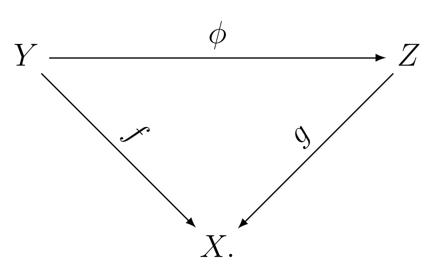
$$\langle 11\rangle A = \langle 4 + 3\sqrt{3}\rangle\langle 4 - 3\sqrt{3}\rangle,$$

or

$$\langle 13\rangle A = \langle 4 + \sqrt{3}\rangle\langle 4 - \sqrt{3}\rangle,$$

(3) *If $p > 3$ and 3 is not a square mod p (e.g $p = 5$ and 7), the ideal $\langle p\rangle$ is prime in A.*

Definition 5.14. *Let $\mathcal{C}$ be a category and let X be an object of $\mathcal{C}$. Let $\mathcal{D} = \mathcal{C}|_X$ be the category whose objects consist of pairs $f\colon Y \longrightarrow X$, where f is a morphism of $\mathcal{C}$, and whose morphisms, consist of commutative diagrams*



$\mathcal{D}$ is known as the category over X. If X is a scheme, then a scheme over X is exactly an object of the category of schemes over X. Let R

be a ring. ***Affine n-space over*** *R, denoted $\mathbb{A}^n_R$, is the spectrum of the polynomial ring $R[x_1, x_2, \dots, x_n]$.*

One of the key ideas of schemes, is to work over arbitrary bases. Note that since there is an inclusion $R \longrightarrow R[x_1, x_2, \dots, x_n]$ of rings, affine space over R is a scheme over $\operatorname{Spec} R$. Thus we may define affine space over any affine scheme. Now we turn to the definition of projective schemes.

The definition mirrors that for affine schemes. First we start with a graded ring S,

$$S = \bigoplus_{d \in \mathbb{N}} S_d.$$

We set

$$S_+ = \bigoplus_{d > 0} S_d,$$

and we let $\operatorname{Proj} S$ denote the set of all homogeneous prime ideals of S, which do not contain S_+. We put a topology on $\operatorname{Proj} S$ analogously to the way we put a topology on $\operatorname{Spec} S$; if $\mathfrak{a}$ is a homogeneous ideal of S, then we set

$$V(\mathfrak{a}) = \{\, \mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{a} \subset \mathfrak{p} \,\}.$$

The Zariski topology is the topology where these are the closed sets. If $\mathfrak{p}$ is a homogeneous prime ideal, then $S_{(\mathfrak{p})}$ denotes the elements of degree zero in the localisation of S at the set of homogenous elements which do not belong to $\mathfrak{p}$. We define a sheaf of rings $\mathcal{O}_X$ on $X = \operatorname{Proj} S$ by considering, for an open set $U \subset X$, all functions

$$s \colon U \longrightarrow \coprod_{\mathfrak{p} \in U} S_{(\mathfrak{p})},$$

such that $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$, which are locally represented by quotients. That is given any point $\mathfrak{q} \in U$, there is an open neighbourhood V of $\mathfrak{p}$ in U and homogeneous elements a and f in S of the same degree, such that for every $\mathfrak{p} \in V$, $f \notin \mathfrak{p}$ and $s(\mathfrak{p})$ is represented by the class of $a/f \in S_{(\mathfrak{p})}$.

Proposition 5.15. *Let S be a graded ring and set $X = \operatorname{Proj} S$.*

1. *For every $\mathfrak{p} \in X$, the stalk $\mathcal{O}_{X,\mathfrak{p}}$ is isomorphic to $S_{(\mathfrak{p})}$.*
2. *For any homogeneous element $f \in S_+$, set*

$$U_f = \{\, \mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p} \,\}.$$

Then U_f is open in $\operatorname{Proj} S$, these sets cover X and we have an isomorphism of locally ringed spaces

$$(U_f, \mathcal{O}_X|_{U_f}) \simeq \operatorname{Spec} S_{(f)}.$$

where $S_{(f)}$ consists of all elements of degree zero in the localisation S_f.

In particular Proj S *is a scheme.*

Proof. The proof of (1) follows similar lines to the affine case and is left as an exercise for the reader. $U_f = X - V(\langle f \rangle)$ and so U_f is certainly open and these sets certainly cover X. We are going to define an isomorphism

$$(g, g^{\#})\colon (U_f, \mathcal{O}_X|_{U_f}) \longrightarrow \operatorname{Spec} S_{(f)}.$$

If $\mathfrak{a}$ is any homogeneous ideal of S, consider the ideal $\mathfrak{a}S_f \cap S_{(f)}$. In particular if $\mathfrak{p}$ is a prime ideal of S, then $\phi(\mathfrak{p}) = \mathfrak{p}S_f \cap S_{(f)}$ is a prime ideal of $S_{(f)}$. It is easy to see that ϕ is a bijection. Now $\mathfrak{a} \subset \mathfrak{p}$ iff

$$\mathfrak{a}S_f \cap S_{(f)} \subset \mathfrak{p}S_f \cap S_{(f)} = \phi(\mathfrak{p}),$$

so that ϕ is a homeomorphism. If $\mathfrak{p} \in U_f$ then $S_{(\mathfrak{p})}$ and $(S_{(f)})_{\phi(\mathfrak{p})}$ are naturally isomorphic. As in the proof of (5.10), this induces a morphism $g^{\#}$ of sheaves which is easily seen to be an isomorphism. □

Definition 5.16. *Let R be a ring.* ***Projective n-space over R****, denoted $\mathbb{P}^n_R$, is the proj of the polynomial ring $R[x_1, x_2, \dots, x_n]$.*

Note that $\mathbb{P}^n_R$ is a scheme over Spec R.

Definition-Lemma 5.17. *If X is a topological space, then let $t(X)$ be the set of irreducible closed subsets of X. Then $t(X)$ is naturally a topological space and if we define a map $\alpha\colon X \longrightarrow t(X)$ by sending a point to its closure then α induces a bijection between the closed sets of X and $t(X)$.*

Proof. Observe that

- If $Y \subset X$ is a closed subset, then $t(Y) \subset t(X)$,
- if Y_1 and Y_2 are two closed subsets, then $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$, and
- if Y_α is any collection of closed subsets, then $t(\cap Y_\alpha) = \cap t(Y_\alpha)$.

The defines a topology on $t(X)$ and the rest is clear. □

Theorem 5.18. *Let k be an algebraically closed field. Then there is a fully faithful functor t from the category of varieties over k to the category of schemes. For any variety V, the set of points of V may be recovered from the closed points of $t(V)$ and the sheaf of regular functions is the restriction of the structure sheaf to the set of closed points.*

Proof. We will show that $(t(V), \alpha_* \mathcal{O}_V)$ is a scheme, where $\mathcal{O}_V$ is the sheaf of regular functions on V. As any variety has an open affine cover, it suffices to prove this for an affine variety, with coordinate ring A. Let X be the spectrum of A. We are going to a define a morphism of locally ringed spaces,

$$\beta = (f, f^{\#})\colon (V, \mathcal{O}_V) \longrightarrow (X, \mathcal{O}_X).$$

If $p \in V$, then let $f(p) = m_p \in X$ be the maximal ideal of elements of A vanishing at p. By the Nullstellensatz, f induces a bijection between the closed points of X and the points of V. It is easy to see that f is a homeomorphism onto its image. Now let $U \subset X$ be an open set. We need to define a ring homomorphism

$$f^{\#}(U)\colon \mathcal{O}_X(U) \longrightarrow f_* \mathcal{O}_V(f^{-1}(U)).$$

Let $s \in \mathcal{O}_X(U)$. We want to define $r = f^{\#}(U)(s)$. Pick $p \in U$. Then we define $r(p)$ to be the image of $s(m_p) \in A_{m_p}$ inside the quotient

$$A_{m_p}/m_p \simeq k.$$

It is easy to see that r is a regular function and that $f^{\#}(U)$ is a ring isomorphism. As the irreducible subsets of V are in bijection with the prime ideals of A, it follows that $(X, \mathcal{O}_X)$ is isomorphic to $(t(V), \alpha_* \mathcal{O}_V)$, and so the latter is an affine scheme.

Note that there is a natural inclusion

$$k \subset A,$$

which associates to a scalar the constant function on V. But then X is a scheme over $\operatorname{Spec} k$. It is easy to check that t is fully faithful. □