4. Sheaves

Definition 4.1. *Let X be a topological space. A **presheaf of groups** $\mathcal{F}$ on X is a a function which assigns to every open set $U \subset X$ a group $\mathcal{F}(U)$ and to every inclusion $V \subset U$ a restriction map,*

$$\rho_{UV} \colon \mathcal{F}(U) \longrightarrow \mathcal{F}(V),$$

which is a group homomorphism, such that if $W \subset V \subset U$, then

$$\rho_{VW} \circ \rho_{UV} = \rho_{UW}.$$

Succintly put, a pre-sheaf is a contravariant functor from $\mathfrak{Top}(X)$ to the category $(\underline{\text{Groups}})$ of groups. Put this way, it is clear what we mean by a presheaf of rings, etc. The elements of $\mathcal{F}(U)$ are called *sections*. We almost always denote $\rho_{UV}(s) = s|_V$. U_{ij} denotes $U_i \cap U_j$.

Example 4.2. *Let X be a topological space and let G be a group. Define a presheaf $\mathcal{G}$ as follows. Let U be any open subset of X. $\mathcal{G}(U)$ is defined to be the set of constant functions from X to G. The restriction maps are the obvious ones.*

Definition 4.3. *A **sheaf** $\mathcal{F}$ on a topological space is a presheaf which satisfies the following two axioms.*

1. *Given an open cover U_i of U an open subset of X, and a collection of sections s_i on U_i, such that $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ then there is a section s on U such that $s|_{U_i} = s_i$.*
2. *Given an open cover U_i of U an open subset of X, if s is a section on U such that $s|_{U_i} = 0$, then s is zero.*

Note that we could easily combine (1) and (2) and require that there is a unique s, which is patched together from the s_i. It is very easy to give lots of examples of sheaves and presheaves. Basically, any collection of functions is a sheaf.

Example 4.4. *Let M be a complex manifold. Then there are a collection of sheaves on M. The sheaf of holomorphic functions, the sheaf of C^∞-functions and the sheaf of continuous functions. In all cases, the restrictions maps are the obvious ones, and there are obvious inclusions of sheaves.*

Given a variety X, the sheaf of regular functions is a sheaf of rings.

Note however that in general the presheaf defined in (4.2) is not a sheaf. For example, take $X = \{a, b\}$ to be the topological space with the discrete topology and take $G = \mathbb{Z}$. Let $U_1 = \{a\}$ and $U_2 = \{b\}$ and suppose $s_1 = 0$ and $s_2 = 1$. Then there is no global constant function which restricts to both 0 and 1.

However this is easily fixed. Take $\mathcal{F}$ to be the sheaf of locally constant functions.

Definition 4.5. *Let X be a topological space and let $\mathcal{F}$ be a presheaf on X. Let $p \in X$. The stalk of $\mathcal{F}$ at p, denoted $\mathcal{F}_p$, is the inverse limit*

$$\lim_{p \in U} \mathcal{F}(U).$$

It is useful to untwist this definition. An element of the stalk is a pair (s, U), such that $s \in \mathcal{F}(U)$, modulo the equivalence relation,

$$(s, U) \sim (t, V)$$

if there is an open subset $W \subset U \cap V$ such that

$$s|_W = t|_V.$$

In other words, we only care about what s looks like in an arbitrarily small neighbourhood of p. Note that when we have a sheaf of rings, the stalk is often a local ring.

Definition-Lemma 4.6. *Let X be an affine variety and let $p \in X$.*

Then the stalk of the structure sheaf of X at p, $\mathcal{O}_{X,p}$ is equal to the localisation of $A(X)$ at the ideal of $p \in X$.

Proof. There is an obvious ring homomorphism

$$A(X) \longrightarrow \mathcal{O}_{X,p}$$

which just sends a polynomial f to the equivalence class (f, X). Suppose that $f \notin m$. Then $p \in U_f \subset X$ and $(1/f, U_f)$ represents the inverse of (f, X) in the ring $\mathcal{O}_{X,p}$. By the universal property of the localisation there is a ring homomorphism

$$A(X)_m \longrightarrow \mathcal{O}_{X,p}.$$

which is clearly injective. Now suppose that we have an element (σ, U) of $\mathcal{O}_{X,p}$. Since sets of the form U_f form a basis for the topology, we may assume that $U = U_g$. By the results of Lecture 3, $\sigma = f/g^n \in A(X)_g \subset A(X)_m$, for some f and n. □

Example 4.7. *Let M be a complex manifold of dimension n and let p be a point of M. Then*

$$\mathcal{O}^h_{M,p} \simeq \mathbb{C}\{z_1, z_2, \dots, z_n\},$$

the ring of convergent power series, since locally about p, M looks like $\mathbb{C}^n$ about zero, and any holomorphic functions is determined by its Taylor series. On the other hand if M is a real manifold of dimension n there is a ring homomorphism

$$\mathcal{C}^\infty_{M,p} \longrightarrow \mathbb{R}\{x_1, x_2, \dots, x_n\},$$

but the kernel is simply huge. In other words, there are lots of infinitely differentiable functions with a trivial Taylor series.

Definition 4.8. *A map between presheaves is a natural transformation of the corresponding functors.*

Untwisting the definition, a map between presheaves

$$f\colon \mathcal{F} \longrightarrow \mathcal{G}$$

assigns to every open set U a group homomorphism

$$f(U)\colon \mathcal{F}(U) \longrightarrow \mathcal{G}(U),$$

such that the following diagram always commutes

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "f(U)"] \arrow[d, "\rho_{UV}"'] & \mathcal{G}(U) \arrow[d, "\sigma_{UV}"'] \\ \mathcal{F}(V) \arrow[r, "f(V)"] & \mathcal{G}(V). \end{tikzcd}$$

Note that this gives us a category of presheaves, together with a full subcategory of sheaves.

Definition-Lemma 4.9. *Let $\mathcal{F}$ be a presheaf.*

Then the ***sheaf associated to the presheaf****, is a sheaf $\mathcal{F}^+$, together with a morphism of sheaves $u\colon \mathcal{F} \longrightarrow \mathcal{F}^+$ which is universal amongst all such morphisms of sheaves: that is given any morphism of presheaves*

$$f\colon \mathcal{F} \longrightarrow \mathcal{G},$$

where $\mathcal{G}$ is a sheaf, there is a unique induced morphism of sheaves which makes the following diagram commute

$$\begin{tikzcd} \mathcal{F} \arrow[r, "f"] \arrow[d, "u"'] & \mathcal{G}. \\ \mathcal{F}^+ \arrow[ur, dashed] & \end{tikzcd}$$

Proof. We just give the construction of $\mathcal{F}^+$ and leave the details to the reader. Let H be the direct sum of all the stalks of $\mathcal{F}$. Let $U \subset X$ be an open set. A section s of $\mathcal{F}^+$ is by definition a function $U \longrightarrow H$ which sends a point p to an element a germ $s(p) = s_p \in \mathcal{F}_p$, which is locally given by a section of $\mathcal{F}$. That is for every $q \in U$, we require that there is an open subset $V \subset U$ containing q, and a section $t \in \mathcal{F}(V)$ such that (t, V) represents s_p in $\mathcal{F}_p$ for all $p \in V$. $\square$

For example the sheaf associated to the presheaf of constant functions to G, is the sheaf of locally constant functions to G.

Proposition 4.10. *Let $\phi\colon \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves.*

Then ϕ is an isomorphism iff the induced map on stalks is always an isomorphism.

Proof. One direction is clear. So suppose that the map on stalks is an isomorphism. It suffices to prove that $\phi(U)\colon \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is an isomorphism, for every open subset $U \subset X$, since then the inverse morphism ϕ is given by setting $\psi(U) = \phi(U)^{-1}$.

We first show that $\phi(U)$ is injective. Let $s \in \mathcal{F}(U)$ and suppose that $\phi(U)(s) = 0$. Then surely $\phi_p(s_p) = 0$, where $s_p = (s, U) \in \mathcal{F}_p$ and $p \in U$ is arbitrary. Since ϕ_p is injective by assumption, it follows that there is an open set $V_p \subset U$ containing p such that $s|_{V_p} = 0$. $\{V_p\}_{p \in U}$ is an open cover of U and as $\mathcal{F}$ is a sheaf, it follows that $s = 0$. Hence $\phi(U)$ is injective, for every U.

Now we show that $\phi(U)$ is surjective. Suppose that $t \in \mathcal{F}(U)$. Since ϕ_p is surjective, for every p, we may find an open set $p \in U_p \subset U$ and a section $s_p \in \mathcal{F}(U_p)$ such that $\phi(U_p)(s_p) = t|_{U_p}$. Pick p and $q \in U$ and set $V = U_p \cap U_q$. Then $\phi(V)(s_p|_V) = \phi(V)(s_q|_V)$. Since $\phi(V)$ is injective, it follows that $s_p|_V = s_q|_V$. As $\mathcal{F}$ is a sheaf, it follows that there is a section $s \in \mathcal{F}(U)$ such that $\phi(U)(s) = t$. But then $\phi(U)$ is surjective. $\square$

Example 4.11. *Let $X = \mathbb{C} - \{0\}$, let $\mathcal{F} = \mathcal{O}_X$, the sheaf of holomorphic functions and let $\mathcal{G} = \mathcal{O}_X^*$, the sheaf of non-zero holomorphic functions.*

There is a natural map

$$\phi\colon \mathcal{F} \longrightarrow \mathcal{G},$$

which just sends a function f to its exponential. Then ϕ is surjective on stalks; this just says that given a non-zero holmorphic function g, then $\log(g)$ makes sense in a small neighbourhood of any point.

On the other hand $\phi(X)$ is not surjective. Indeed $z \in \mathcal{F}(X)$ is a function which is not in the image, since $\log(z)$ is not globally single valued.

Definition 4.12. *Let $f\colon X \longrightarrow Y$ be a continuous map of topological spaces. Let $\mathcal{F}$ be a sheaf on X. The **pushforward of** $\mathcal{F}$, denoted $f_*\mathcal{F}$, is defined as follows*

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)),$$

where $U \subset Y$ is an open set.

Let $\mathcal{G}$ be a sheaf on Y. The ***inverse image of*** *$\mathcal{G}$, denoted $f^{-1}\mathcal{G}$, is the sheaf assigned to the presheaf*

$$U \longrightarrow \lim_{f(U) \subset V} \mathcal{G}(V),$$

where U is an open subset of X and V ranges over all open subets of Y which contain $f(U)$.

Definition 4.13. *A pair $(X, \mathcal{O}_X)$ is called a* ***ringed space****, if X is a topological space, and $\mathcal{O}_X$ is a sheaf of rings. A morphism $\phi\colon X \longrightarrow Y$ of ringed spaces is a pair $(f, f^{\#})$, consisting of a continuous function $f\colon X \longrightarrow Y$ and a sheaf morphism $f^{\#}\colon \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$.*

A ***locally ringed space****, is a ringed space $(X, \mathcal{O}_X)$ such that in addition every stalk $\mathcal{O}_{X,x}$ of the structure sheaf is a local ring. A morphism of locally ringed spaces is a morphism of ringed spaces, such that for every point $x \in X$, the induced map*

$$f^{\#}_x\colon \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x},$$

is a morphism of local rings (that is the inverse image of the maximal ideal of $\mathcal{O}_{X,x}$ is a subset of the maximal ideal of $\mathcal{O}_{Y,y}$, where $y = f(x)$).

Note that we get a category of ringed spaces, whose objects are ringed spaces and whose morphisms are morphisms of ringed spaces. Further the category of locally ringed spaces is a subcategory.

Definition 4.14. *Let $(X, \mathcal{O}_X)$ be a ringed space. An* ***$\mathcal{O}_X$-module*** *is a sheaf $\mathcal{F}$ such that for every open set $U \subset X$, $\mathcal{F}(U)$ has the structure of an $\mathcal{O}_X(U)$-module, compatible with the restriction map, in an obvious way.*

Using (4.9) we may define various natural operations on sheaves. For example, let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{O}_X$-modules. The tensor product of $\mathcal{F}$ and $\mathcal{G}$, denoted $\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{G}$, is the sheaf associated to the presheaf

$$U \longrightarrow \mathcal{F}(U) \underset{\mathcal{O}_X(U)}{\otimes} \mathcal{G}(U),$$

and curly hom, denoted $\mathbf{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, is the sheaf associated to the presheaf

$$U \longrightarrow \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U)).$$

Let $f\colon \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. The kernel of f is the sheaf which assigns to every open set U the kernel of the homomorphism $f(U)\colon \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$. Similarly the image is the sheaf associated to the presheaf which assigns to every open set U the image of the homomorphism $f(U)\colon \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$. We say that ϕ is injective iff $\mathrm{Ker}(\phi) = 0$ and we say that ϕ is surjective iff $\mathrm{Im}(\phi) = \mathcal{G}$.

Given a morphism of ringed spaces, and a sheaf $\mathcal{G}$ of $\mathcal{O}_X$-modules, the pullback of $\mathcal{G}$, denoted $\phi^*\mathcal{G}$, is the sheaf of $\mathcal{O}_Y$-modules,

$$\phi^{-1}\mathcal{G} \underset{f^{-1}\mathcal{O}_X}{\otimes} \mathcal{O}_Y.$$