3. Some commutative algebra

Definition 3.1. Let \( R \) be a ring. We say that \( R \) is graded, if there is a direct sum decomposition,

\[
R = \bigoplus_{n \in \mathbb{N}} R_n,
\]

where each \( R_n \) is an additive subgroup of \( R \), such that

\[
R_d R_e \subset R_{d+e}.
\]

The elements of \( R_d \) are called the homogeneous elements of order \( d \).

Let \( R \) be a graded ring. We say that an \( R \)-module \( M \) is graded if there is a direct sum decomposition

\[
M = \bigoplus_{n \in \mathbb{N}} M_n,
\]

compatible with the grading on \( R \) in the obvious way,

\[
R_d M_n \subset M_{d+n}.
\]

A morphism of graded modules is an \( R \)-module map \( \phi: M \rightarrow N \) of graded modules, which respects the grading,

\[
\phi(M_n) \subset N_n.
\]

A graded submodule is a submodule for which the inclusion map is a graded morphism. A graded ideal \( I \) of \( R \) is an ideal, which when considered as a submodule is a graded submodule.

Note that the kernel and cokernel of a morphism of graded modules is a graded module. Note also that an ideal is a graded ideal iff it is generated by homogeneous elements. Here is the key example.

Example 3.2. Let \( R \) be the polynomial ring over a ring \( S \). Define a direct sum decomposition of \( R \) by taking \( R_n \) to be the set of homogeneous polynomials of degree \( n \). Given a graded ideal \( I \) in \( R \), that is an ideal generated by homogeneous elements of \( R \), the quotient is a graded ring.

We will also need the notion of localisation, which is a straightforward generalisation of the notion of the field of fractions.

Definition 3.3. Let \( R \) be a ring. We say that a subset \( S \) of \( R \) is multiplicatively closed if for every \( s_1 \) and \( s_2 \) in \( S \), \( s_1 s_2 \in S \), that is

\[
S \cdot S \subset S.
\]
Definition-Lemma 3.4. Let $R$ be a ring and let $S$ be a multiplicatively
closed subset. The **localisation of $R$ at $S$**, denoted $R_S$, is a ring $R_S$
together with a ring homomorphism

$$
\phi: R \rightarrow R_S,
$$

with the property that for every $s \in S$, $\phi(s)$ is a unit in $R_S$ which
is universal amongst all such rings. That is given any morphism

$$
\psi: R \rightarrow R'
$$

with the property that $\psi(s)$ is a unit, for every $s \in S$, there is a unique
ring homomorphism

![Diagram]

**Proof.** This is almost identical to the construction of the field of frac-
tions, and so we will skip most of the details. Formally we define $R_S$
to be the set of all pairs $(r, s)$, where $r \in R$ and $s \in S$, modulo the
equivalence relation,

$$(r_1, s_1) \sim (r_2, s_2) \iff s(r_1 s_2 - r_2 s_1) \text{ for some } s \in S.$$

We denote an equivalence class by $[r, s]$ (or more informally by $r/s$).
Addition and multiplication is defined in the obvious way. \hfill \Box

Note that $R$ is an integral domain, then $S = R - \{0\}$ is multi-

plicatively closed and the localisation is precisely the field of fractions.
Note that as we are not assuming that $S$ is an integral domain, we need
to throw in the extra factor of $s$, in the definition of the equivalence relation
and the natural map $R \rightarrow R_S$ is not injective.

Remark 3.5. Suppose that $R$ is a graded ring, and that $S$ is a multi-

plicative set, generated by homogeneous elements. Then $R_S$ is a graded
ring, where the grading is given by

$$
\deg f/g = \deg f - \deg g,
$$

where, of course, the grading is now given by the integers.

Here are two key examples.

Example 3.6. Suppose that $p$ is a prime ideal in a ring $R$. Then

$S = R - p$ is a multiplicatively closed subset of $R$. The localisation is
denoted $R_p$. It elements consist of all fractions $r/f$, where $f \notin p$. On
the other hand, suppose that \( f \in R \) is not nilpotent. Then the set of powers of \( f \),
\[
S = \{ f^n \mid n \in \mathbb{N} \},
\]
is a multiplicatively closed subset. The localisation consists of all elements of the form \( r/f^n \).

For example, take \( R = \mathbb{Z} \) and \( f = 2 \). Then \( R_f = \mathbb{Z}[1/2] \subset \mathbb{Q} \) consists of all fractions whose denominator is a power of two.

Note that if \( A \) is a finitely generated \( K \)-algebra, without nilpotents, and \( f \in A \), then \( A_f \) is also a finitely generated \( K \)-algebra.

**Definition 3.7.** Let \( R \) be a ring. We say that \( R \) is a **local ring** if there is a unique maximal ideal.

\( R_p \) is a local ring. The ideal generated by \( p \) is the unique maximal ideal.

Recall Hilbert’s Nullstellensatz:

**Theorem 3.8** (Nullstellensatz). Let \( I \trianglelefteq k[X] \) be an ideal, where \( k \) is an algebraically closed field.
Then \( I(V(I)) = \sqrt{I} \).

One very well known consequence of (3.8) (which is sometimes also called the Nullstellensatz) is that the maximal ideals of \( k[X] \) are in correspondence with the points of \( \mathbb{A}^n \). By virtue of the Nullstellensatz we have:

**Theorem 3.9.** Let \( k \) be an algebraically closed field.
Then there is an equivalence of categories between the category of varieties over an \( k \) and the opposite category of the category of finitely generated \( k \)-algebras, without nilpotents.

**Proof.** We define a functor \( A \) as follows. Given an affine variety \( X \), let \( A(X) \) be the \( k \)-algebra of regular functions from \( X \) to \( k \). In particular \( A(X) \) does not contain any nilpotent elements. Given an embedding of \( X \) into affine space \( \mathbb{A}^n \), \( A(X) \) is simply the quotient of the corresponding polynomial ring \( k[X_1, X_2, \ldots, X_n] \) by \( I(X) \). Thus \( A(X) \) is a finitely generated \( k \)-algebra. Given a morphism \( f : X \rightarrow Y \), we get a map \( A(f) : A(Y) \rightarrow A(X) \), by composition. If \( \phi : Y \rightarrow K \) is a regular function, an element of \( A(Y) \), then \( A(f)(\phi) = \phi \circ f \). In particular \( A \) is a contravariant functor.

We have to show that \( A \) is fully faithful and essentially surjective. Let \( f : X \rightarrow Y \) be a morphism of affine varieties and let \( F = A(f) : A(Y) \rightarrow A(X) \) be the corresponding \( k \)-algebra homomorphism. If \( x \in X \) is a point then \( F^{-1}(m_x) = m_y \), where \( y = f(x) \).
and \( m_x \subset A \) is the maximal ideal of functions vanishing at \( x \). It follows that we can recover \( f \) from \( F = A(f) \), so that the functor \( A \) is faithful.

Now suppose that we are given a \( k \)-algebra homomorphism \( F: A(Y) \rightarrow A(X) \). Pick an embedding of \( Y \subset \mathbb{A}^n \) into affine space. Let \( y_1, y_2, \ldots, y_n \) be coordinates on \( \mathbb{A}^n \) and \( v_1, v_2, \ldots, v_n \in A(Y) \) be the restriction to \( Y \) of the corresponding regular functions. Then we get regular functions \( f_i = F(v_i) \in A(X) \). We can define a morphism \( f: X \rightarrow \mathbb{A}^n \) by the rule \( f(x) = (f_1(x), f_2(x), \ldots, f_n(x)) \). It is straightforward to check that the image of \( f \) lands inside \( Y \) and that \( A(f) = F \). Thus \( A \) is fully faithful.

Let \( A \) be any finitely generated \( k \)-algebra. If we pick generators \( \xi_1, \xi_2, \ldots, \xi_n \), then we get a surjective \( k \)-algebra surjective homomorphism

\[
\mathbb{K}[X_1, X_2, \ldots, X_k] \rightarrow A,
\]

simply by sending \( X_i \) to \( \xi_i \) and applying the universal property of the polynomial ring. Let \( I \) be the kernel, and let \( X = V(I) \). Since \( A \) does not have any nilpotents \( I \) is a radical ideal, and by the Nullstellensatz it follows that \( A(X) \simeq A \). Thus the functor \( A \) is essentially surjective. \( \square \)

**Definition 3.10.** Let \( X \) be a quasi-projective variety and let \( f \) be a regular function on \( X \).

\[
U_f = \{ x \in X \mid f(x) \neq 0 \}.
\]

Note that \( U_f \) is open. In fact if \( X \) is affine then these sets form a base for the Zariski topology.

**Lemma 3.11.** Let \( X \) be an affine variety, and let \( f \in A = A(X) \), the coordinate ring of \( X \).

Then \( U_f \) is an affine variety whose coordinate ring is naturally isomorphic to \( A_f \).

**Proof.** Suppose that \( X \subset \mathbb{A}^n \), and let \( f_1, f_2, \ldots, f_k \) be generators of the ideal of \( X \), so that \( I(X) = \langle f_1, f_2, \ldots, f_k \rangle \). Let \( Y \subset \mathbb{A}^{n+1} \) be the zero set of \( f_1, f_2, \ldots, f_k \) and \( g(x, y) = f(x)y - 1 \), where \( y \) is the variable corresponding to the extra last coordinate, and \( x = x_1, x_2, \ldots, x_n \). Now \( Y \) is an affine variety and there is a natural projection map

\[
\pi: Y \rightarrow X,
\]

given by sending \((x_1, x_2, \ldots, x_n, y) \) to \((x_1, x_2, \ldots, x_n) \). The image of \( \pi \) is precisely \( U_f \). On the other hand, there is a natural morphism the other way,

\[
\psi: U_f \rightarrow Y,
\]

given by sending \((x_1, x_2, \ldots, x_n) \) to \((x_1, x_2, \ldots, x_n, 1/f) \). Clearly the composition either way is the identity, so that \( Y \) and \( U_f \) are isomorphic.
Consider the quotient
\[ K[x_1, x_2, \ldots, x_n, y]/\langle f_1, f_2, \ldots, f_k, f(x)y - 1 \rangle. \]
This is equal to
\[ A[y]/\langle f(x)y - 1 \rangle. \]
There is a natural ring homomorphism
\[ A \longrightarrow A[y]/\langle f(x)y - 1 \rangle, \]
which is the composition of the natural inclusion and the natural projection. Now the image of \( f \) is invertible. Thus, by the universal property of the localisation, there is an induced ring homomorphism
\[ A_f \longrightarrow A[y]/\langle f(x)y - 1 \rangle. \]
It is easy to see that this map is bijective, whence it is an isomorphism.

It remains to see that this map is bijective, whence it is an isomorphism. It is easy to see that \( I = \langle f_1, f_2, \ldots, f_k, f(x)y - 1 \rangle \) is the ideal of \( Y \). Since \( Y = V(I) \), it suffices to prove that \( I \) is radical. As an ideal \( I \) is radical iff the quotient has no nilpotents, by the isomorphism we have already established, it suffices to prove that \( A_f \) has no nilpotents. So suppose that \( g/f^m \) is a nilpotent element of \( A_f \). Then \( f^n g \) is a nilpotent for some \( n \). It follows that \((fg)^n = 0 \). As \( A \) has no nilpotents it follows that \( fg = 0 \), so that \( g/f^m = 0 \), to begin with. \( \square \)

We will also need some multilinear algebra. We recall the definition of the tensor product.

**Definition 3.12.** Let \( M \) and \( N \) be two \( R \)-modules. The **tensor product** of \( M \) and \( N \), is an \( R \)-module \( M \otimes_R N \), together with a bilinear map
\[ u: M \times N \longrightarrow M \otimes_R N, \]
which is universal amongst all such bilinear maps. That is given any bilinear map
\[ M \times N \longrightarrow P \]
there is a unique induced map \( M \otimes_R N \longrightarrow P \) which makes the following diagram commute
\[ \begin{array}{ccc}
M \times N & \longrightarrow & M \otimes_R N \\
& u & \\
& & P
\end{array} \]
The image of \((m,n)\) is denoted \(m \otimes n\). Note that every element of \(M \otimes R N\) is a linear combination

\[\sum_i r_i m_i \otimes n_i.\]

Note the following elementary (and natural) isomorphisms,

\[(M \oplus N) \otimes P \simeq M \otimes P \oplus N \otimes P\]

Thus if \(M\) is a finitely generated free module, so that \(M\) is isomorphic to \(R^n\), the direct sum of \(n\) copies of \(R\), then \(R^n \otimes R^m \simeq R^{mn}\).

In fact we can write down an explicit set of free generators. If \(e_1, e_2, \ldots, e_m\) and \(f_1, f_2, \ldots, f_n\) are the free generators of \(R^n\) and \(R^m\), it follows that \(e_i \otimes e_j\) are free generators of the tensor product.

**Definition 3.13.** Let \(M\) be an \(R\)-module. Then the **symmetric product of \(M\) with itself \(k\) times** is an \(R\)-module \(\text{Sym}^k(M)\) together with a symmetric bilinear map

\[M^k \longrightarrow \text{Sym}^k(M)\]

which is universal amongst all such symmetric bilinear maps.

If \(M\) is freely generated by \(e_1, e_2, \ldots, e_n\) and \(k = 2\), then a basis for \(\text{Sym}^2(M)\) is given \(e_i e_j\) (where of course \(e_j e_i = e_i e_j\)).

**Definition 3.14.** Let \(V\) and \(W\) be vector spaces over \(K\). A **pairing** between \(V\) and \(W\) is a bilinear map

\[V \times W \longrightarrow K\]

We say that a pairing is **perfect**, if for every \(v \in V\) there is a \(w \in W\) such that \(\langle v, w \rangle \neq 0\), and vice-versa.

**Lemma 3.15.** Let \(V\) and \(W\) be finite dimensional vector spaces over \(K\). Then the following pieces of data are equivalent

1. an isomorphism \(W \longrightarrow V^*\),
2. a perfect pairing \(V \times W \longrightarrow K\).

**Proof.** There is an obvious perfect pairing between \(V\) and \(V^*\), given by

\[\langle v, \phi \rangle = \phi(v).\]

Composing this with the obvious map

\[V \times W \longrightarrow V \times V^*,\]
we see that (1) implies (2). Now suppose we are given a pairing $V \times W \to K$. Given $w \in W$, we get a linear map $\phi: V \to K$ by sending $v$ to $\phi(v) = \langle v, w \rangle$. It is easy to see that this gives us a linear map $W \subset V^*$ and since we have a perfect pairing, this map is injective. In particular the dimension of $W$ is less than the dimension of $V$. By symmetry we have a reverse inclusion. Thus $V$ and $W$ have the same dimension and so the map $W \to V^*$ is in fact an isomorphism $\square$

**Lemma 3.16.** Let $V$ and $W$ be two finite dimensional vector spaces. Then there are natural isomorphisms,

1. $(V \otimes W)^* \simeq V^* \otimes W^*$.
2. $(\text{Sym}^d V)^* \simeq \text{Sym}^d V^*$.
3. $(\bigwedge^d V)^* \simeq \bigwedge^d V^*$.

**Proof.** We first prove (1). It suffices to exhibit a perfect pairing

$$V \otimes W \times V^* \otimes W^* \to K$$

Given $\phi \in V^*$ and $\psi \in W^*$, define a bilinear map

$$V \times W \to K$$

by sending

$$(v, w) \to \phi(v)\psi(w).$$

By the universal property of the tensor product this induces a linear map

$$V \otimes W \to K$$

Extending linearly, we get a pairing, which is easily checked to be a perfect pairing. Hence (1).

By an obvious induction we get that

$$(V^\otimes d)^* = (V^*)^\otimes d.$$ 

It is easy to check that this induces perfect pairings between the symmetric and alternating tensors. $\square$

**Lemma 3.17.** Let $V$ be a vector space. Then we have a natural isomorphism of graded rings,

$$K[V] = \bigoplus_{d \in \mathbb{N}} \text{Sym}^d(V^*)$$

**Proof.** It suffices to prove that

$$\text{Sym}^d(V^*)$$

is naturally isomorphic to the set of homogeneous polynomials of degree $d$ on $V$. First the case $n = 1$. In this case $\text{Sym}^1(V^*) = V^*$ and we are looking at linear functionals on $V$. If we choose a basis for $V$ and let
$X_1, X_2, \ldots, X_n$ be the dual basis for $V^*$, then we get a natural basis for the space of polynomials of degree one.

Taking the natural basis for $(\text{Sym}^d(V))^* = \text{Sym}^d(V^*)$, we get a basis with all monomials of degree $d$. □

Recall that if we replace symmetric by skew-symmetric, then we get the wedge product $\bigwedge^d M$. If $M$ is free with generators $e_1, e_2, \ldots, e_k$, then $\bigwedge^d M$ is also free, with generators $e_I$, where $I$ runs over set of increasing indices $e_1 < e_2 < \cdots < e_d$, and $e_I$ denotes the element $e_{i_1} \wedge e_{i_2} \wedge \ldots e_{i_d}$.

Note that if we have a vector space $V$ over a field $K$, one of the most important invariants of a tensor is its rank.

**Definition 3.18.** Let $\sigma \in V^\otimes d$. The rank $k$ of $\sigma$ is the smallest number $k$, such that we may write $\sigma$ as a sum of $k$ primitive tensors, that is tensors of the form $v_1 \otimes v_2 \otimes \cdots \otimes v_d$.

Note that given a tensor $\sigma \in V^\otimes 2$ and a basis $e_1, e_2, \ldots, e_n$ for $V$, if we expand $\sigma$ in terms of the induced basis for $V^\otimes 2$, then we get a $n \times n$ matrix. For example, suppose that we are given a vector space of dimension two with basis $\{e, f\}$. Then a general tensor is of the form

$$\sigma = ae \otimes e + be \otimes f + cf \otimes e + df \otimes f.$$ 

The associated matrix is then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

**Lemma 3.19.** Let $\sigma \in V^\otimes 2$.

Then the rank of $\sigma$ is equal to the rank of the associated matrix.

**Proof.** Note that an $n \times n$ matrix $A$ has rank at most $k$ iff it is the product of an $k \times n$ matrix $B$ by an $n \times k$ matrix $C$. If $v_i$ denotes the columns of $B$ and $w_j$ denotes the rows of $C$, then

$$\sum v_i \otimes w_i = \left( \sum_{k} b_{ki}e_k \right) \otimes \left( \sum_{l} c_{il}e_l \right)$$

$$= \sum_{i,k,l} b_{ki}c_{il}e_k \otimes e_l$$

$$= \sum_{k,l} a_{kl}e_k \otimes e_l = \sigma.$$ □
Perhaps the easiest case to visualise, is the case of rank one. If a matrix has rank one, then it is the product of a column vector $v$ times a row vector $w$. Then $\sigma = v \otimes w$. Note that we can talk about the rank of a symmetric or alternating tensor.