We recall the definition of a category:

**Definition 2.1.** A category $\mathcal{C}$ is the data of two collections. The first collection is called the **objects** of $\mathcal{C}$ and is denoted $\text{Obj}(\mathcal{C})$. Given two objects $X$ and $Y$ of $\mathcal{C}$, we associate another collection $\text{Hom}(X,Y)$, called the **morphisms** between $X$ and $Y$. Further we are given a law of composition for morphisms: given three objects $X, Y$ and $Z$, there is an assignment

$$\text{Hom}(X,Y) \times \text{Hom}(Y,Z) \rightarrow \text{Hom}(X,Z).$$

Given two morphisms, $f \in \text{Hom}(X,Y)$ and $g \in \text{Hom}(Y,Z)$, $g \circ f \in \text{Hom}(X,Z)$ denotes the composition. Further this data satisfies the following axioms:

1. Composition is associative,
   $$h \circ (g \circ f) = (h \circ g) \circ f,$$
   for all objects $X, Y, Z, W$ and all morphisms $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and $h : Z \rightarrow W$.

2. For every object $X$, there is a special morphism $i = i_X \in \text{Hom}(X,X)$ which acts as an identity under composition. That is for all $f \in \text{Hom}(X,Y)$,
   $$f \circ i_X = f = i_Y \circ f.$$

We say that a category $\mathcal{C}$ is **small** if for every pair of objects $X$ and $Y$, $\text{Hom}(X,Y)$ is a set.

There are an abundance of categories.

**Example 2.2.** The category $(\text{Sets})$ of sets and functions; the category of $(\text{Groups})$ groups and group homomorphisms; the category $(\text{Vec})$ of vector spaces and linear maps; the category $(\text{Top})$ of topological spaces and continuous maps; the category $(\text{Rings})$ of rings and ring homomorphisms.

Let $X$ be a topological space. We can define a category $\text{Top} X$ associated to $X$ as follows. The objects of $\text{Top} X$ are simply the open subsets of $X$. Given two open subsets $U$ and $V$,

$$\text{Hom}(U,V) = \begin{cases} i_{UV} & \text{if } U \subseteq V \\ \emptyset & \text{otherwise} \end{cases}.$$

Here $i_{UV}$ is a formal symbol. Composition of morphisms is defined in the obvious way (in fact the definition is forced, there are no choices to be made).
Definition 2.3. We say that a category $\mathcal{D}$ is a subcategory of $\mathcal{C}$ if every object of $\mathcal{D}$ is an object of $\mathcal{C}$ and for every pair of objects $X$ and $Y$ of $\mathcal{D}$, $\text{Hom}_\mathcal{D}(X,Y)$ is a subset of $\text{Hom}_\mathcal{C}(X,Y)$ (that is every morphism in $\mathcal{D}$ is a morphism in $\mathcal{C}$).

We say that $\mathcal{D}$ is a full subcategory of $\mathcal{C}$, if for every pair of objects $X$ and $Y$ of $\mathcal{D}$, $\text{Hom}_\mathcal{D}(X,Y)$ is equal to $\text{Hom}_\mathcal{C}(X,Y)$.

The category of finite sets is a full subcategory of the category (Sets) of sets. Similarly the category of finite dimensional linear spaces is a full subcategory of the category (Vec) of vector spaces. By comparison the category (Groups) of groups is a subcategory of the category (Sets) of sets, but it is not a full subcategory. In other words not every function is a group homomorphism.

It is easy construct new categories from old ones:

Definition 2.4. Given a category $\mathcal{C}$, the opposite category, denoted $\mathcal{C}^{\text{op}}$, is the category, whose objects are the same as $\mathcal{C}$, but whose morphisms go the other way, so that

$$\text{Hom}_{\mathcal{C}^{\text{op}}}(X,Y) = \text{Hom}_\mathcal{C}(Y,X).$$

Definition 2.5. The inverse of a morphism $f: X \to Y$ is a morphism $g: Y \to X$, such that $f \circ g$ and $g \circ f$ are both the identity map. If the inverse of $f$ exists, then we say that $f$ is an isomorphism and that $X$ and $Y$ are isomorphic.

Definition 2.6. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A covariant functor from $\mathcal{C}$ to $\mathcal{D}$ assigns to every object $X$ of $\mathcal{C}$ an object $\text{F}(X)$ of $\mathcal{D}$ and to every morphism $f: X \to Y$ in $\mathcal{C}$ a morphism $\text{F}(f): \text{F}(X) \to \text{F}(Y)$ in $\mathcal{C}$, compatible with composition and the identity.

That is

$$\text{F}(g \circ f) = \text{F}(g) \circ \text{F}(f) \quad \text{and} \quad \text{F}(i_X) = i_{\text{F}(X)}.$$  

A contravariant functor $\text{F}$ is the same as covariant functor, except that arrows are reversed,

$$\text{F}(f): \text{F}(Y) \to \text{F}(X),$$

and

$$\text{F}(g \circ f) = \text{F}(f) \circ \text{F}(g).$$

In other words a contravariant functor $\text{F}: \mathcal{C} \to \mathcal{D}$ is the same as a covariant functor $\text{F}: \mathcal{C}^{\text{op}} \to \mathcal{D}$

It is easy to give examples of functors. Let

$$\text{F}: (\text{Rings}) \to (\text{Groups}),$$
be the functor which assigns to every ring $R$, the underlying additive
group, and to every ring homomorphism $f$, the corresponding group
homomorphism (the same map of course).

It is easy to check that $F$ is indeed a functor; for obvious reasons it
is called a forgetful functor and there are many such functors.

Note that we may compose functors in the obvious way and that
there is an identity functor. Slightly more interestingly there is an
obvious contravariant functor from a category to its opposite.

There are three non-trivial well-known functors. First there is a
functor, denoted $H_*$, from the category (Top) of topological spaces
to the category of (graded) groups, which assigns to every topological
space its singular homology. Similarly there is a contravariant functor
from category (Top) of topological spaces to the category of (graded)
rings, which assigns to every topological space its singular cohomology.

The second and third are much more general.

**Definition 2.7.** Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. We say that $F$ is
faithful if for every $f$ and $g$, morphisms in $\mathcal{C}$, $F(f) = F(g)$ iff $f = g$.
We say that $F$ is full if for every morphism $h : F(X) \rightarrow F(Y)$ in
$\mathcal{D}$, there is a morphism $f$ in $\mathcal{C}$ such that $F(f) = h$. We say that $F$ is
essentially surjective if for every object $A$ in $\mathcal{D}$ there is an object
$X$ in $\mathcal{C}$ such that $A$ is isomorphic to $F(X)$.

We say that $F$ is an equivalence of categories if $F$ is fully faithful and
essentially surjective.

For example, let $\mathcal{D}$ be the category of finite dimensional vector spaces
over a field $K$. Let $\mathcal{C}$ be the category whose objects are the natural
numbers, and such that the set of morphisms between two natural
numbers $m$ and $n$, is equal to the set of $m \times n$ matrices, with the
obvious rule of composition. Then $\mathcal{C}$ is naturally a full subcategory of
$\mathcal{D}$ (assign to $n$ the “standard” vector space $K^n$) and the inclusion map,
considered as a functor, is an equivalence of categories. Note however
that there is no functor the other way.

More generally, given a category $\mathcal{D}$, one may form a quotient category
$\mathcal{C}$. Informally the objects and morphisms of $\mathcal{C}$ are equivalence classes
of objects of $\mathcal{D}$, under isomorphism.

We defer the third definition. One of the more interesting notions of
category theory, is the theory of limits.

**Definition 2.8.** Let $I$ be a category and let $F : I \rightarrow \mathcal{C}$ be a func-
tor. A pre-limit for $F$ is an object $L$ of $\mathcal{C}$, together with morphisms
$f_i : F(I) \rightarrow L$, for every object $I$ of $I$, which are compatible in the fol-
lowing sense: Given a morphism $f : I \rightarrow J$ in $I$, the following diagram
The direct limit of $F$, denoted $L = \operatorname{lim}_I F$ is a pre-limit $L$, which is universal amongst all pre-limits in the following sense: Given any pre-limit $L'$ there is a unique morphism $g: L \rightarrow L'$, such that for every object $I$ in $I$, the following diagram commutes:

$$
\begin{array}{ccc}
F(I) & \xrightarrow{F(f)} & F(J) \\
\downarrow f_I & & \downarrow f_J \\
L & \xrightarrow{g} & L'.
\end{array}
$$

Informally, then, if we think of a pre-limit as being to the right of every object $F(I)$, then the limit is the furthest pre-limit to the left. Note that limits, if they exist at all, are unique, up to unique isomorphism, by the standard argument. Note also that there is a dual notion, the notion of inverse limits. In this case, $F$ is a contravariant functor and all the arrows go the other way (informally, then, a pre-limit is to the left of every object $F(I)$ and a limit is any pre-limit which is furthest to the right).

Let us look at some special cases. First take the category with one object and one morphism. In this case a functor picks out an object. It is clear that in this case the limit is the same object.

At the other extreme one can take the identity functor, so that $I = \mathcal{C}$. A direct limit, if it exists at all, is an object to which all other objects map (in a compatible fashion). Recall the notion of a terminal object. A terminal object has the property that every object has a unique map to it. In the case that a category has a terminal object, then the direct limit of the identity functor is the terminal object. The category (Sets) of sets has as terminal object any set with one object; the category (Vec) of vector spaces any space of dimension zero.

Dually, an indirect limit, if it exists at all, is an object which maps to all other objects. Recall the notion of an initial object. An initial object has the property that for every object in the category, there is a unique map from the initial object. In the case that a category has
an initial object, then the indirect limit of the identity functor is the
initial object. The empty set is an initial object of the category (Sets)
of sets; the group with one element is an initial object in the category
(Groups) of groups.

Now take as category two objects, with two morphisms (that is the
two identity maps). A functor picks out two objects, call them X and
Y. First consider the case of the direct limit. A prelimit is the data
of an object Z, together with a pair or morphisms, \( f: X \to Z \) and
\( g: Y \to Z \). This pre-limit is a limit iff it is universal amongst all such
pre-limits. That is suppose we are given two morphisms \( f': X \to Z' \)
and \( g': Y \to Z' \), then there is a unique induced morphism \( h: Z \to Z' \),
such that the following diagram commutes

\[
\begin{array}{ccc}
X & \xrightarrow{f'} & Z' \\
\downarrow{g'} & & \\
Y & \xleftarrow{g} & Z
\end{array}
\]

Dually, consider the case of a indirect limit, where all the arrows
are reversed. A prelimit is the data of an object Z, together with a
pair or morphisms, \( f: Z \to X \) and \( g: Z \to Y \). This pre-limit is a
limit iff it is universal amongst all such pre-limits. That is suppose we
are given two morphisms \( f': Z' \to X \) and \( g': Z' \to Y \), then there
is a unique induced morphism \( h: Z' \to Z \), such that the following
diagram commutes

\[
\begin{array}{ccc}
X & \xleftarrow{f} & Z' \\
\downarrow{g} & & \\
Y & \xrightarrow{g'} & Z
\end{array}
\]
Definition 2.9. Let $X$ and $Y$ be two objects of a category $C$. The direct product (respectively the indirect product) of $X$ and $Y$, also known as the direct sum (respectively product), is the direct (respectively indirect) limit of the (respectively contravariant) functor above.

The direct sum of two sets is their disjoint union; similarly for topological spaces; the direct sum of two vector spaces is the ordinary direct sum; similarly for groups and rings. The direct product of two sets is the ordinary cartesian product; the direct product of two topological spaces is the product of the spaces and so on. Note that for groups, rings and vector spaces, the coincidence that the direct sum and product are in fact isomorphic.

Now let us be a little more ambitious. Take a category with three objects and five morphisms. The two non-trivial morphisms should have the same domain, but different targets.

Definition 2.10. Suppose we are given a diagram

$$
\begin{array}{ccc}
    & Y & \\
    & \downarrow{g} & \\
X & \downarrow{f} & B.
\end{array}
$$

The direct limit of the corresponding functor, denoted $X \times_B Y$, is known as the fibre product or fibre square.

As with the definition of the direct product, there is an accompanying commutative diagram

$$
\begin{array}{ccc}
    & Z' & \\
    & \downarrow{Z} & \\
    & Y & \\
    & \downarrow{g} & \\
X & \downarrow{f} & B.
\end{array}
$$

Note that if $B$ is a terminal object, then the fibre product is nothing more than a direct product.

Lemma 2.11. The category $(\text{Sets})$ of sets admits fibre products.
Proof. It is easy to check that
\[ X \times_Y B = \{ (x, y) \in X \times Y \mid f(x) = g(y) \}, \]
does the trick. \qed

The fibre product is sometimes also known as the pullback. In other words we think of the morphism
\[ X \times_Y B \rightarrow X, \]
as the pullback of the map \( g: Y \rightarrow B \) along the map \( f: X \rightarrow B \). In particular the fibre of the former map over the point \( x \in X \) is equal to the fibre of the map \( g \) over the point \( f(x) \).

The dual notion is that of pushout. Basically take the diagram above, flip about the \( Y - X \)-diagonal and reverse the arrows. Thus if we start with the diagram
\[
\begin{array}{ccc}
B & \xrightarrow{g} & Y \\
\downarrow{f} & & \downarrow \\
X & & \\
\end{array}
\]

the pushout \( Z \) has enjoys the universal property encoded in the following commutative diagram:

\[
\begin{array}{cccc}
B & \xrightarrow{g} & Y & \xrightarrow{m} Z \\
\downarrow{f} & & \downarrow & \\
X & & Z' & \\
\end{array}
\]

For example, consider the category of rings. Suppose we are given two ring homomorphisms \( A \rightarrow B \) and \( A \rightarrow C \), and two ring homomorphisms \( B \rightarrow P \) and \( C \rightarrow P \). Then we get a bilinear map \( B \times C \rightarrow P \), using multiplication in \( P \). It is then easy to see that the pushout is the tensor product \( B \otimes_C A \).
We now turn to the third important functor. We first note that given two categories \( C \) and \( D \), the collection of all functors from \( C \) to \( D \) is a category, denoted \( \text{Fun}(C, D) \). The objects of this category are simply functors from \( C \) to \( D \). Given two functors \( F \) and \( G \), a morphism between them is a natural transformation:

**Definition 2.12.** Let \( F \) and \( G \) be two functors from a category \( C \) to a category \( D \). A **natural transformation** \( u \) from \( F \) to \( G \) assigns to every object \( X \) of \( C \) a morphism \( u_X : F(X) \to G(X) \) such that for every morphism \( f : X \to Y \) in \( C \) the following diagram commutes

\[
\begin{array}{ccc}
F(X) & \xrightarrow{F(f)} & F(Y) \\
\downarrow u_X & & \downarrow u_Y \\
G(X) & \xrightarrow{G(f)} & G(Y). 
\end{array}
\]

It is easy to check that we may compose natural transformations, that this composition is associative and that the natural transformation which assigns to every object \( X \), the identity map from \( F(X) \) to \( F(X) \) acts as an identity, so that \( \text{Fun}(C, D) \) is indeed a category.

Suppose that \( C \) is a small category. Let \( Y \) be an object of \( C \). I claim that we get a functor \( h_Y : C \to \text{(Sets)} \). Given an object \( X \) of \( C \), we associate the set \( h_Y(X) = \text{Hom}(X, Y) \). Given a morphism \( f : X \to X' \), note that we get a map

\[
h_Y(f) : \text{Hom}(X', Y) \to \text{Hom}(X, Y),
\]

which takes a morphism \( g \) and assigns the morphism \( h_Y(f)(g) = g \circ f \). It is easy to check that \( h_Y \) is a contravariant functor. On the other hand, varying \( Y \), I claim we get a functor

\[
h : C \to \text{Fun}(\text{C}^{\text{op}}, \text{(Sets)})
\]

At the level of objects, the definition of this functor is obvious. Given \( Y \in C \) we assign the object \( h_Y \in \text{Fun}(\text{C}^{\text{op}}, \text{(Sets)}) \). On the other hand, given a morphism \( f : Y \to Y' \), I claim that we get a natural transformation \( h(f) \) between the two functors \( h_Y \) and \( h_{Y'} \), going from \( \text{C}^{\text{op}} \) to \( \text{(Sets)} \). Thus given an object \( X \) in \( C \), we are given a morphism

\[
h(f)_X : h_Y(X) = \text{Hom}(X, Y) \to h_{Y'}(X) = \text{Hom}(X, Y').
\]

The definition of \( h(f)_X \) is clear. Given \( g \in \text{Hom}(X, Y) \), send this to \( h(f)_X(g) = f \circ g \). It is easy to check that \( h(f) \) is indeed a natural transformation and that \( h \) is a functor. More significantly:
**Theorem 2.13** (Yoneda’s Lemma). *h is fully faithful.*

The proof is left as an exercise for the reader. Yoneda’s Lemma thus says that if we want to understand the category $\mathcal{C}$, we can think of it as a subcategory of the category of contravariant functors from $\mathcal{C}$ to the category $(\text{Sets})$ of sets. For example, suppose that we want to check if $W$ is the fibre product of $f: X \rightarrow Z$ and $g: X \rightarrow Z$, given that it is at least a pre-limit. If $\mathcal{C}$ were the category $(\text{Sets})$ of sets this is completely straightforward. Using Yoneda’s Lemma, one can reduce to this case. Indeed pick any object $V$ of $\mathcal{C}$. If $W$ is the fibre product, then one might hope that $h_V(W)$ is the pushout of $h_V(f): h_V(Z) \rightarrow h_V(X)$ and $h_V(g): h_V(Z) \rightarrow h_V(Y)$ in the category $(\text{Sets})$ of sets. Yoneda’s Lemma says that if this holds for every object $V$, then in fact $W$ is the fibre product in $\mathcal{C}$.

In these terms obviously the most fundamental question is to ask which of these functors is in the image:

**Definition 2.14.** We say that the functor $F: \mathcal{C}^{\text{op}} \rightarrow (\text{Sets})$ is *representable* (by $Y$) if it is isomorphic to $h_Y$, for some object $Y$ of $\mathcal{C}$.

By Yoneda’s Lemma, if $F$ is representable by $Y$ then $Y$ is determined up to unique isomorphism.