15. The functor of points and the Hilbert scheme

Clearly a scheme contains much more information than the topology of the underlying set. Nevertheless it is possible to consider a scheme as a hierarchy of sets of points.

Now in many categories, it is possible to recover the underlying set of points by considering the set of all possible morphisms from a fixed object Z,

$$|X| = \operatorname{Hom}_{\mathcal{C}}(Z, X).$$

For example

(1) In the category of topological spaces, take Z to be a point.
(2) In the category of groups, take $Z = \mathbb{Z}$.
(3) In the category of rings, take $Z = \mathbb{Z}[x]$.

For schemes, it will surely not work to take a single object. We need to consider

$$\operatorname{Hom}_{\mathcal{C}}(Z, X),$$

for every possible choice of Z. In other words, given an object X, we should consider the functor

$$h_X : \mathcal{C} \longrightarrow \mathcal{D} \qquad \text{given by} \qquad h_X(Y) = \operatorname{Hom}_{\mathcal{C}}(Y, X).$$

where $\mathcal{C}$ is the category of schemes and $\mathcal{D}$ is the category of sets.

For example take

$$X = \operatorname{Spec} R \qquad \text{where} \qquad R = \mathbb{Z}[x_1, x_2, \dots, x_n]/\langle f_1, f_2, \dots, f_k \rangle.$$

In this case, if T is any other ring, then

$$h_X(T) := h_X(\operatorname{Spec} T),$$

is the set of ring homomorphisms

$$R \longrightarrow T.$$

Any such ring homomorphism is determined by the images $a_1, a_2, \dots, a_n$ of the $x_1, x_2, \dots, x_n$. Further $a_1, a_2, \dots, a_n$ must be a solution to the equations $f_i = 0$. Thus

$$h_X(T) = \{\, (a_1, a_2, \dots, a_n) \mid f_i(a_1, a_2, \dots, a_n) = 0, \forall i \,\}.$$

Of course, as before, we can raise the level of generality one more step, and consider the functor

$$h : \mathcal{C} \longrightarrow \mathbf{Hom}(\mathcal{C}^\circ, \mathcal{D}) \qquad \text{which sends} \qquad X \longrightarrow h_X.$$

Recall:

Lemma 15.1 (Yoneda's Lemma)**.** *Let $\mathcal{C}$ be a category and let X and X' be any two objects of $\mathcal{C}$.*

(1) *If F is any functor from $\mathcal{C}$ to the category of sets, the natural transformations between h_X and F are in natural correspondence with the elements of $F(X)$.*
(2) *h is an equivalence of categories with a full subcategory of $\mathbf{Hom}(\mathcal{C}^\circ, \mathcal{D})$.*

Proof. Given a natural transformation

$$\alpha\colon h_X \longrightarrow F,$$

we assign the element $\alpha(i_X)$, where $i_X\colon X \longrightarrow X$ is the identity map. The inverse takes $p \in F(X)$ to the map α sending $f \in \operatorname{Hom}(Y, X)$ to $F(f)(p) \in F(Y)$.

(2) then follows by applying (1) to the functor $F = \operatorname{Hom}(\_, Y)$. □

One very useful fact, is that in the category of schemes we can do a little better than this.

Proposition 15.2. *If R is a ring, then there is an equivalence of functors*

$$h\colon \mathcal{C} \longrightarrow \mathbf{Hom}(\mathcal{R}, \mathcal{D}),$$

where $\mathcal{C}$ is now the category of R-schemes (that is schemes over $\operatorname{Spec} R$) and $\mathcal{R}$ is the category of R-algebras.

Proof. Let $S = \operatorname{Spec} R$. Denote by h_X the functor

$$\operatorname{Hom}(\_, X),$$

on the category of affine S-schemes. It is enough to prove that any natural transformation

$$\phi\colon h_X \longrightarrow h_{X'},$$

comes from a unique morphism f over S from X to X'. Let U_α be an open affine cover of X and apply ϕ to $U_\alpha \subset X$ to get morphisms $U_\alpha \longrightarrow X'$. These morphisms agree on overlaps, so that we get the morphism f. □

Put differently, we only need to consider all morphisms from affine schemes into X.

Definition 15.3. *Let F be a functor from the category of k-algebras to sets. Let $p \in F(K)$ (note that if $F = h_X$, then p is a k-rational point of X).*

The ***Zariski tangent space*** *to F at p is*

$$F(k[\epsilon]/\langle \epsilon^2 \rangle).$$

Of course this just describes the Zariski tangent space as a set. To get multiplication by a scalar, consider the map

$$k[\epsilon]/\langle \epsilon^2 \rangle \longrightarrow k[\epsilon]/\langle \epsilon^2 \rangle \qquad \text{where} \qquad \epsilon \longrightarrow a\epsilon.$$

Since this is a k-algebra homomorphism, it follows that we get an induced map on the Zariski tangent space. To define addition of vectors, we have to assume that F preserves fibre products. In this case, note that

$$k[\epsilon, \epsilon']/\langle \epsilon, \epsilon' \rangle^2.$$

is the tensor product of $k[\epsilon]/\langle \epsilon^2 \rangle$ with itself. Note that there is a natural map

$$k[\epsilon]/\langle \epsilon^2 \rangle \longrightarrow k[\epsilon, \epsilon']/\langle \epsilon, \epsilon' \rangle^2 \qquad \text{where} \qquad \epsilon \longrightarrow \epsilon + \epsilon'.$$

If F preserves fibre products, this induces a map

$$T_pF \times T_pF \longrightarrow T_pF,$$

which gives us addition of vectors.

Let us look at some examples:

Proposition 15.4. *The functor F which assigns to every ring T the set of all quotients of T^r which are isomorphic to the free T-module of rank one is represented by $\mathbb{P}^r_{\mathbb{Z}}$.*

Note that it is important at this level of generality to work with quotients of dimension one and not submodules of dimension one. For example, we really would not want to consider multiplication by 2,

$$\times 2 \colon \mathbb{Z} \longrightarrow \mathbb{Z},$$

as defining a point of $\mathbb{P}^1_{\mathbb{Z}}$. More generally we have

Proposition 15.5. *Let A be a ring. The functor F which assigns to every A-algebra T the set of all A-algebra quotients of T^r which are isomorphic to the free T-module of rank k is represented by a scheme $\mathbb{G}_A(r-k, r)$.*

Definition 15.6. *Let $X \subset \mathbb{P}^r_k$ be a projective scheme. The* ***Hilbert polynomial*** *is the unique polynomial such that*

$$P(m) = H^0(X, \mathcal{O}_X(m)),$$

for all m sufficiently large.

Definition 15.7. *Let S be a scheme, and let n be a positive integer. the functor F assigns to every S-scheme T, the set*

$$\{\, X \subset \mathbb{P}^n_T \mid X \textit{ is projective and flat over } T \,\}.$$

It is not hard to prove:

Theorem 15.8. *Let $X \subset \mathbb{P}^n_T$ be a projective variety over a an integral scheme T.*

Then X is flat over T if and only if the Hilbert polynomial is constant.

It is important to realise that it is crucial that T is integral.

The big result, due to Grothendieck is then:

Theorem 15.9. *The functor F above is representable. If we fix a polynomial P, then the subfunctor F_P of F consisting of those families with Hilbert polynomial P is a projective subscheme of the Hilbert scheme.*

The corresponding scheme is called the Hilbert scheme. For example, consider plane curves of degree d. The component of the Hilbert scheme is particularly nice in these examples, it is just represented by a projective space of dimension

$$\binom{d+2}{2} - 1.$$

Note that (15.9) says something quite amazing. Every flat family of closed subschemes of projective space is induced by pulling back from the Hilbert scheme.

There are two crucial ingredients to the proof of (15.9). The first is relative easy to prove:

Proposition 15.10. *Fix a polynomial P and a field k. Then there is an integer n_0 such that for every $X \subset \mathbb{P}^r_k$ with Hilbert polynomial P and for every $n \geq n_0$ we have*

(1) $h^i(X, \mathcal{O}_X(n)) = 0$*, for* $i > 0$,
(2) $\mathcal{O}_X(n)$ *is globally generated,*
(3) $P(n) = h^0(X, \mathcal{O}_X(n))$.

The second result is quite deep:

Theorem 15.11. *Let $\pi \colon X \longrightarrow S$ be a projective morphism.*

Then there is a scheme T, the disjoint union of locally closed subsets of S such that $\pi_T \colon X_T \longrightarrow T$ is flat and if $S' \longrightarrow S$ is any morphism such that $\pi_{S'} \colon X_{S'} \longrightarrow S'$ is flat, then there is an induced morphism $S' \longrightarrow T$.