14. Flatness

In this section we go deeper into the meaning of what a flat limit is, and what it means for a family of schemes to vary continuously.

Definition 14.1. *Let X be an equi-dimensional scheme of dimension n. We say that X is* ***regular*** *at $x \in X$ if the Zariski tangent space at x is of dimension n.*

Suppose first that we have a regular one dimensional scheme S. Let $s \in S$ be a point and suppose that we are given a family over $S^* = S - \{s\}$, $\pi\colon X^* \longrightarrow S^*$. We suppose that our family is given as a closed embedding in $\mathbb{A}^n_U$. Then the only sensible thing to do is to take the closure of X^* in $\mathbb{A}^n_S$, to get a family $\pi\colon X \longrightarrow S$.

To make this more concrete, suppose that $S = \operatorname{Spec} A$ is affine and that $t \in A$ generates the maximal ideal of the point $s \in S$. In this case $S^* = U_t = \operatorname{Spec} A[t^{-1}]$ and $I(X^*) \subset A[t^{-1}][x_1, x_2, \dots, x_n]$. But then

$$I(X) = I(X^*) \cap A[x_1, x_2, \dots, x_n].$$

Even more concretely, suppose that $A = k[t]$, so that $S = \mathbb{A}^1_k$. In this case $S^* = \operatorname{Spec} k[t, t^{-1}]$ and so

$$I(X) = I(X^*) \cap k[t][x_1, x_2, \dots, x_n].$$

It follows then that the ideal of X_0 is the ideal generated by the vector subspace V consisting of all elements of $k[x_1, x_2, \dots, x_n]$ which are the limits of elements of $I(X_t)$.

Note that the flat limit depends very much on the choice of embedding. For example, consider the two families over the punctured affine line, given by the ideals

$$\langle x^2 - 1\rangle \qquad \text{and} \qquad \langle x^2 - t^{-2}\rangle.$$

In the first family the central fibre consists of two reduced points. In the second, we have to multiply through by t^2, to get $t^2x^2 - 1$, which has limit the empty set. Clearly both families we start with are isomorphic.

Now let us consider a much more complicated and interesting example. Suppose that we start with three collinear lines in $\mathbb{A}^3_k$, whose Zariski tangent space spans $\mathbb{A}^3_k$. Any such three lines are isomorphic to the three coordinate axes. We imagine fixing two lines and rotating the third (fixing the triple point of intersection).

Specifically, we may suppose that our three lines $L \cup M \cup N$ are given by

$$\langle y, z\rangle \qquad \langle x, z\rangle \qquad \text{and} \qquad \langle x, y\rangle.$$

We rotate the third line, to get a family of lines N_t given by

$$\langle x - y, z - tx\rangle.$$

Thus we get a family of schemes

$$X_t = L \cup M \cup N_t,$$

and we wish to determine the limit

$$X_0.$$

The trick is to observe that the ideal of the three coordinate axes is

$$\langle xy, xz, yz \rangle.$$

Thus the ideal of X_t, $t \neq 0$, is of the form

$$\langle Q_1, Q_2, Q_3 \rangle, ,$$

where

$$\begin{aligned} Q_1 &= z(z - tx), \\ Q_2 &= z(z - ty), \\ Q_3 &= (z - tx)(z - ty). \end{aligned}$$

Now if we simply set $t = 0$, then in all three cases we get z^2. Thus if we take a difference, something interesting happens:

$$Q_1 - Q_3 = tyz - t^2xy \qquad \text{and} \qquad Q_2 - Q_3 = txz - t^2xy.$$

For $t \neq 0$, we may divide through by t, so that

$$\frac{Q_1 - Q_3}{t} = yz - txy \qquad \text{and} \qquad \frac{Q_2 - Q_3}{t} = xz - txy.$$

It follows that the ideal of X_0 contains z^2, xz and yz. Clearly, we still need something else; since if $z = 0$ then all three terms vanish and so the plane $z = 0$ is contained in the support, which is too much. If we consider the difference

$$x\left(\frac{Q_1 - Q_3}{t}\right) - y\left(\frac{Q_2 - Q_3}{t}\right) = txy(x - y).$$

Thus the ideal of X_0 contains $xy(x - y)$. Putting all this together,

$$I(X_0) \supset \langle xz, yz, z^2, xy(x - y) \rangle.$$

The claim is that this is the ideal of the limit, so that we get equality. The interesting thing to realise is that this is not the ideal of the union of the three limiting lines. Indeed the ideal of the union is

$$I(L \cup M \cup N_0) = \langle z, xy(x - y) \rangle.$$

Thus

$$I(X_0) = I(L \cup M \cup N_0) \cap \langle x, y, z \rangle^2.$$

In other words, X_0 has an embedded point at the origin. Once we realise this, it is not in fact hard to prove that we do indeed get the indicated ideal. The key point, as observed before, is that the Zariski tangent space to the three lines, when $t \neq 0$, is three dimensional. Since the dimension of the Zariski tangent space is upper semi-continuous, along any section over B, it follows that the dimension of the Zariski tangent space to the limit has dimension at least three (and hence dimension three).

But the union of the three lines has Zariski tangent space contained in the plane $z = 0$. It follows that the limiting scheme must have an embedded point, and this embedded point must stick out of the plane. It is easy then to prove that

$$I(X_0) \subset I(L \cup M \cup N_0) \cap \langle x, y, z \rangle^2,$$

and we are done. As useful as these constructions of the flat limit are in practice, we are still lacking a complete theoretical understanding of what is going on.

In particular, it would be nice to have a notion that is independent of the embedding and which works over non-algebraically closed fields, or at least which works over a base of dimension greater than one. There is such a notion, but it is a little hard to take in at first.

Definition 14.2. *We say that an R-module M is flat, if whenever we have a short exact sequence of R-modules,*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

the sequence remains exact after tensoring with M,

$$0 \longrightarrow A \underset{R}{\otimes} M \longrightarrow B \underset{R}{\otimes} M \longrightarrow C \underset{R}{\otimes} M \longrightarrow 0.$$

Lemma 14.3. *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C,$$

be an exact sequence of R-modules, and let M be an R-module.

Then the following sequence

$$0 \longrightarrow \operatorname{Hom}_R(M, A) \longrightarrow \operatorname{Hom}_R(M, B) \longrightarrow \operatorname{Hom}_R(M, C),$$

is exact.

Similarly let

$$A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

be an exact sequence of R-modules, and let M be an R-module.

Then the following sequence

$$\operatorname{Hom}_R(A, M) \longrightarrow \operatorname{Hom}_R(B, M) \longrightarrow \operatorname{Hom}_R(C, M) \longrightarrow 0,$$

is exact.

Proof. Let $f\colon M \longrightarrow A$ be a non-zero R-module homomorphism. Then $a = f(m) \neq 0$, for some $m \in M$. But then the image b of a is not equal to zero, and so if $g \in \operatorname{Hom}_R(M, B)$ denotes the image of f, we have $g(m) = b \neq 0$, so that we have exactness at $\operatorname{Hom}_R(M, A)$.

Now consider exactness at $\operatorname{Hom}_R(M, B)$. Since the composition of $A \longrightarrow B$ and $B \longrightarrow C$ is zero, it is easy to see that the composition of $\operatorname{Hom}_R(M, A) \longrightarrow \operatorname{Hom}_R(M, B)$ and $\operatorname{Hom}_R(M, B) \longrightarrow \operatorname{Hom}_R(M, C)$ is zero. To finish, suppose that $g \in \operatorname{Hom}_R(MB)$ is sent to zero. This means, that for every $m \in M$, $b = g(m)$ is sent to zero in C. By exactness of the first sequence, there is a unique element $a \in A$ sent to b. Define a map $f\colon M \longrightarrow A$ by $f(m) = a$. It is easy to check that f is an R-module homomorphism.

The second sequence is checked just as easily. □

Definition 14.4. *Two functors $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ and $G\colon \mathcal{D} \longrightarrow \mathcal{C}$ are said to be* ***adjoint*** *if there is a natural bijection between $\operatorname{Hom}_{\mathcal{C}}(F(X), Y)$ and $\operatorname{Hom}_{\mathcal{D}}(X, G(Y))$. We say that F is a* ***left adjoint of*** *G and G is a* ***right adjoint of*** *F.*

Here natural, means that the bijections induce a natural transformation between the two induced functors,

$$\mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{S},$$

where the last is the category of sets, and the two functors are

$$(X, Y) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(F(X), Y) \qquad \text{and} \qquad (X, Y) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(X, G(Y)).$$

Lemma 14.5. *Fix an R-module N. The two functors,*

$$F\colon \mathcal{R} \longrightarrow \mathcal{R} \qquad \textit{given by} \qquad F(M) = M \underset{R}{\otimes} N,$$

and

$$G\colon \mathcal{R} \longrightarrow \mathcal{R} \qquad \textit{given by} \qquad G(P) = \operatorname{Hom}_R(N, P),$$

are adjoint.

Proof. It suffices to exhibit a natural bijection between

$$\operatorname{Hom}(M \otimes N, P) \qquad \text{and} \qquad \operatorname{Hom}(M, \operatorname{Hom}(N, P)).$$

But by the universal property of the tensor product the first set is equal to the set of bilinear maps

$$M \times N \longrightarrow P.$$

Given a bilinear map $\phi(m, n)$, we get a ring homomorphism

$$M \longrightarrow \operatorname{Hom}(N, P)$$

simply by sending m to the ring homomorphism $n \longrightarrow f(m, n)$. The induced bijection is clearly natural. □

Lemma 14.6. *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

be a short exact sequence of R-modules, and let M be any R-module.

Then the following sequence is exact,

$$A \underset{R}{\otimes} M \longrightarrow B \underset{R}{\otimes} M \longrightarrow C \underset{R}{\otimes} M \longrightarrow 0.$$

Proof. By (14.3) the sequence

$$\operatorname{Hom}_R(A, \operatorname{Hom}_R(M, P)) \longrightarrow \operatorname{Hom}_R(B, \operatorname{Hom}_R(M, P)) \longrightarrow \operatorname{Hom}_R(C, \operatorname{Hom}_R(M, P)) \longrightarrow 0,$$

is exact. By adjointness, it follows that the sequence

$$\operatorname{Hom}_R(A \otimes M, P) \longrightarrow \operatorname{Hom}_R(B \otimes M, P) \longrightarrow \operatorname{Hom}_R(B \otimes M, P) \longrightarrow 0,$$

is exact. But then it is easy to see that

$$A \underset{R}{\otimes} M \longrightarrow B \underset{R}{\otimes} M \longrightarrow C \underset{R}{\otimes} M \longrightarrow 0,$$

is exact. □

In particular, an R-module M is flat iff for every inculusion

$$A \longrightarrow B \qquad \text{then} \qquad A \underset{R}{\otimes} M \longrightarrow B \underset{R}{\otimes} M,$$

is also an inclusion. Since every free module is a direct sum of copies of R and tensoring by R has no effect, it follows that every free module is flat. In particular every module over a field (aka a vector space) is flat. More generally every module over a Dedekind domain (eg a principal ideal domain) M is flat iff it is torsion-free.

Definition 14.7. *A morphism $\pi \colon X \longrightarrow B$ is **flat** if every local ring $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{B,b}$-module, where $b = \pi(x)$.*

Lemma 14.8. *Let $B = \operatorname{Spec} A$ be a regular one-dimensional scheme. Let $) \in B$ be a closed point and let $B^* = B - \{0\}$. Let $X \subset \mathbb{A}^n_B$. be a closed subscheme, with projection $\pi \colon X \longrightarrow B$. The following are equivalent:*

1. *π is flat over 0.*
2. *The fibre $X_0 = \pi^{-1}(0)$ is the limit of the fibres X_b, as $b \to 0$.*
3. *No irreducible or embedded component is supported on X_0.*

Proof. Let $X^* = \pi^1(B^*) \subset X$. As X is closed, it contains the closure of X^*. Thus (2) is equivalent to the statement that X is the closure of X^*. On the other, hand

$$X = \bar{X}^* \cup X_0,$$

is a non-trivial decomposition of X iff (3) fails. Thus (2) and (3) are equivalent.

To see the equivalence of (1) and (3), we need the following piece of commutative algebra; $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{B,0}$-module for every $x \in X_0$ iff $\mathcal{O}_X(X)$ is a torsion free R-module. □

Lemma 14.9. *Flatness is preserved under base change.*

Proof. Let $\pi\colon X \longrightarrow Y$ be a flat morphism and let $Y' \longrightarrow Y$ be an arbitrary base change. Since flatness is a local condition, we reduce immediately to the case when all three schemes are affine. Using the equivalence of categories, we are reduced to proving that if M is a flat A-module, and A' is an arbitrary A-algebra, then $M' = B \underset{A}{\otimes} A'$ is a flat A'-module, which is a known result in algebra, see (9.1.b) of Hartshorne. □

Lemma 14.10. *Let $X \subset \mathbb{A}^n_B$, where B is a reduced scheme over a field k and let $b \in B$ be a closed point.*

Then π is flat over 0 iff for every morphism $B' \longrightarrow B$, where B' is a regular one dimension scheme, the fibre square is flat over the inverse image of b.

Note that one direction follows from (14.9). The other direction takes quite a bit more work.

Example 14.11. *Let S be a smooth quasi-projective surface and let $\pi\colon T \longrightarrow S$ be the blow up of a point $p \in S$. Then π is not flat. Indeed, if we pick any smooth curve B downstairs passing through p and consider the base change $B \longrightarrow S$ then we the resulting morphism $C \longrightarrow B$ has fibre over p the whole exceptional divisor and fibre over any other point a single point. Now apply* (14.8) *and* (14.10)*.*

Corollary 14.12. *Let $\pi\colon X \longrightarrow Y$ be a flat morphism, where Y is a variety.*

Then π has equi-dimensional fibres.

Proof. By base change, we reduce to the case where Y is integral affine. Cutting by hyperplanes, and taking the normalisation, we reduce to the case where Y is a smooth curve. In this case, the result is clear, by (14.8). □

As useful and appealing as (14.10) is, however it should be realised that flat families over non-reduced bases are also very interesting and don't really fall under the auspices of (14.10). Indeed, suppose we consider flat families over

$$T = \operatorname{Spec} k[\epsilon]/\langle \epsilon^2 \rangle.$$

In this case there is only point, but flatness still turns out to be a highly non-trivial and interesting condition.