

### 13. SHEAF COHOMOLOGY

**Definition 13.1.** *Let  $X$  be a topological space. For every  $i \geq 0$  there are functors  $H^i$  from the category of sheaves of abelian groups on  $X$  to the category of abelian groups such that*

- (1)  $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$ .
- (2) *Given a short exact sequence,*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0,$$

*there are boundary maps*

$$H^i(X, \mathcal{H}) \longrightarrow H^{i+1}(X, \mathcal{F}).$$

*which can be strung together to get a long exact sequence of cohomology.*

In short, sheaf cohomology was invented to fix the lack of exactness, and in fact this property essentially fixes the definition.

**Example 13.2.** *If  $X$  is a simplicial complex (or a CW-complex) then  $H^i(X, \mathbb{Z})$  agrees with the usual definition. The same goes for any other coefficient ring (considered as a local free sheaf).*

*Like ordinary cohomology, sheaf cohomology inherits a cup product,*

$$H^i(X, \mathcal{F}) \otimes H^j(X, \mathcal{G}) \longrightarrow H^{i+j}(X, \mathcal{F} \otimes \mathcal{G}),$$

*where  $(X, \mathcal{O}_X)$  is a ringed space and  $\mathcal{F}$  and  $\mathcal{G}$  are  $\mathcal{O}_X$ -modules. In particular if  $X$  is a projective scheme over  $A$  then*

$$H^i(X, \mathcal{F}),$$

*is an  $A$ -module, where  $\mathcal{F}$  is an  $\mathcal{O}_X$ -module, since  $A = H^0(X, \mathcal{O}_X)$ . In particular if  $A$  is a field, then*

$$H^i(X, \mathcal{F}),$$

*are vector spaces.*

$$h^i(X, \mathcal{F}),$$

*denotes their dimension.*

We would like to have a definition of these groups which allows us to compute. Let  $X$  be a topological space and let  $\mathcal{U} = \{U_i\}$  be an open cover, which is locally finite. The group of  $k$ -cochains is

$$C^k(\mathcal{U}, \mathcal{F}) = \bigoplus_I \Gamma(U_I, \mathcal{F}),$$

where  $I$  runs over all  $(k+1)$ -tuples of indices.

Define a coboundary map

$$\delta^k: C^k(\mathcal{U}, \mathcal{F}) \longrightarrow C^{k+1}(\mathcal{U}, \mathcal{F}).$$

Given  $\sigma = (\sigma_I)$ , we have to construct  $\tau = \delta(\sigma) \in C^{k+1}(\mathcal{U}, \mathcal{F})$ . We just need to determine the components  $\tau_J$  of  $\tau$ . Now  $J = \{i_0, i_2, \dots, i_k\}$ . If we drop an index, then we get a  $k$ -tuple.

$$\tau_J = \left( \sum_{i=0}^k (-1)^i \sigma_{J-\{i_i\}} \right) |_{U_J}.$$

The key point is that  $\delta^2 = 0$ . So we can take cohomology

$$H^i(\mathcal{U}, \mathcal{F}) = Z^i(\mathcal{U}, \mathcal{F}) / B^i(\mathcal{U}, \mathcal{F}).$$

Here  $Z^i$  denotes the group of  $i$ -cocycles, those elements killed by  $\delta^i$  and  $B^i$  denotes the group of coboundaries, those cochains which are in the image of  $\delta^{i-1}$ . Note that  $\delta^i(B^i) = \delta^i \delta^{i-1}(C^{i-1}) = 0$ , so that  $B^i \subset Z^i$ .

The problem is that this is not enough. Perhaps our open cover is not fine enough to capture all the interesting cohomology. A **refinement** of the open cover  $\mathcal{U}$  is an open cover  $\mathcal{V}$ , together with a map  $h$  between the indexing sets, such that if  $V_j$  is an open subset of the refinement, then for the index  $i = h(j)$  such that  $V_j \subset U_i$ . It is straightforward to check that there are maps,

$$H^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(\mathcal{V}, \mathcal{F}),$$

on cohomology. Taking the (direct) limit, we get the Čech cohomology groups,

$$\check{H}^i(X, \mathcal{F}).$$

For example, consider the case  $i = 0$ . Given a cover, a cochain is just a collection of sections,  $(\sigma_i)$ ,  $\sigma_i \in \Gamma(U_i, \mathcal{F})$ . This cochain is a cocycle if  $(\sigma_i - \sigma_j)|_{U_{ij}} = 0$  for every  $i$  and  $j$ . By the sheaf axiom, this means that there is a global section  $\sigma \in \Gamma(X, \mathcal{F})$ , so that in fact

$$H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F}).$$

It is also sometimes possible to untwist the definition of  $H^1$ . A 1-cocycle is precisely the data of a collection

$$(\sigma_{ij}) \in \Gamma(\mathcal{U}, \mathcal{F}),$$

such that

$$\sigma_{ij} - \sigma_{ik} + \sigma_{jk} = 0,$$

where  $\#\{i, j, k\} = 3$ . Note also that cocycles are skew-commutative, for example  $\sigma_{ji} = -\sigma_{ij}$ .

In general of course, one does not want to compute these things using limits. The question is how fine does the cover have to be to compute the cohomology? As a first guess one might require that

$$H^i(U_j, \mathcal{F}) = 0,$$

for all  $j$ , and  $i > 0$ . In other words there is no cohomology on each open subset. But this is not enough. One needs instead the slightly stronger condition that

$$H^i(U_I, \mathcal{F}) = 0.$$

**Theorem 13.3** (Leray). *If  $X$  is a topological space and  $\mathcal{F}$  is a sheaf of abelian groups and  $\mathcal{U}$  is an open cover such that*

$$H^i(U_I, \mathcal{F}) = 0,$$

*for all  $i > 0$  and indices  $I$ , then in fact the natural map*

$$H^i(\mathcal{U}, \mathcal{F}) \simeq \check{H}^i(X, \mathcal{F}),$$

*is an isomorphism.*

It is in fact not too hard to prove:

**Theorem 13.4** (Serre). *Let  $X$  be a noetherian scheme. TFAE*

- (1)  *$X$  is affine,*
- (2)  *$H^i(X, \mathcal{F}) = 0$  for all  $i > 0$  and all quasi-coherent sheaves,*
- (3)  *$H^1(X, \mathcal{I}) = 0$  for all coherent ideals  $\mathcal{I}$ .*

Finally, we need to construct the coboundary maps. Suppose that we are given a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

We want to define

$$H^i(X, \mathcal{H}) \longrightarrow H^{i+1}(X, \mathcal{F}).$$

Cheating a little, we may assume that we have a commutative diagram with exact rows,

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^i(\mathcal{U}, \mathcal{F}) & \longrightarrow & C^i(\mathcal{U}, \mathcal{G}) & \longrightarrow & C^i(\mathcal{U}, \mathcal{H}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C^{i+1}(\mathcal{U}, \mathcal{F}) & \longrightarrow & C^{i+1}(\mathcal{U}, \mathcal{G}) & \longrightarrow & C^{i+1}(\mathcal{U}, \mathcal{H}) \longrightarrow 0. \end{array}$$

Suppose we start with an element  $t \in H^i(X, \mathcal{H})$ . Then  $t$  is the image of  $t' \in H^i(\mathcal{U}, \mathcal{H})$ , for some open cover  $\mathcal{U}$ . In turn  $t'$  is represented by  $\tau \in Z^i(\mathcal{U}, \mathcal{H})$ . Now we may suppose our cover is sufficiently fine, so that  $\tau_I \in \Gamma(U_I, \mathcal{H})$  is the image of  $\sigma_I \in \Gamma(U_I, \mathcal{G})$  (and this fixes the cheat). Applying the boundary map, we get  $\delta(\sigma) \in C^{i+1}(\mathcal{U}, \mathcal{G})$ . Now the image of  $\delta(\sigma)$  in  $C^{i+1}(\mathcal{U}, \mathcal{H})$  is the same as  $\delta(\tau)$ , which is zero, as  $\tau$  is a cocycle. But then by exactness of the bottom rows, we get  $\rho \in C^{i+1}(\mathcal{U}, \mathcal{F})$ . It is straightforward to check that  $\rho$  is a cocycle,

so that we get an element  $r' \in H^{i+1}(\mathcal{U}, \mathcal{F})$ , whence an element  $r$  of  $H^{i+1}(X, \mathcal{F})$ , and that  $r$  does not depend on the choice of  $\sigma$ .

Thus sheaf cohomology does exist (at least when  $X$  is paracompact, which is not a problem for schemes). Let us calculate the cohomology of projective space.

**Theorem 13.5.** *Let  $A$  be a noetherian ring. Let  $X = \mathbb{P}_A^r$ .*

(1) *The natural map  $S \rightarrow \Gamma_*(X, \mathcal{O}_X)$  is an isomorphism.*

(2)

$$H^i(X, \mathcal{O}_X(n)) = 0 \quad \text{for all } 0 < i < r \text{ and } n.$$

(3)

$$H^r(X, \mathcal{O}_X(-r-1)) \simeq A.$$

(4) *The natural map*

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \rightarrow H^r(X, \mathcal{O}_X(-r-1)) \simeq A,$$

*is a perfect pairing of finitely generated free  $A$ -modules.*

*Proof.* Let

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n).$$

Then  $\mathcal{F}$  is a quasi-coherent sheaf. Let  $\mathcal{U}$  be the standard open affine cover. As every intersection is affine, it follows that we may compute using this cover. Now

$$\Gamma(U_I, \mathcal{F}) = S_{x_I},$$

where

$$x_I = \prod_{i \in I} x_i.$$

Thus Čech cohomology is the cohomology of the complex

$$\prod_{i=0}^r S_{x_i} \rightarrow \prod_{i < j}^r S_{x_i x_j} \rightarrow \dots \rightarrow S_{x_0 x_1 \dots x_r}.$$

The kernel of the first map is just  $H^0(X, \mathcal{F})$ , which we already know is  $S$ . Now let us turn to  $H^r(X, \mathcal{F})$ . It is the cokernel of the map

$$\prod_i S_{x_0 x_1 \dots \hat{x}_i \dots x_r} \rightarrow S_{x_0 x_1 \dots x_r}.$$

The last term is the free  $A$ -module with generators all monomials in the Laurent ring (that is we allow both positive and negative powers).

The image is the set of monomials where  $x_i$  has non-negative exponent. Thus the cokernel is naturally identified with the free  $A$ -module generated by arbitrary products of reciprocals  $x_i^{-1}$ ,

$$\{x_0^{l_0} x_1^{l_1} \dots x_r^{l_r} \mid l_i < 0\}.$$

The grading is then given by

$$l = \sum_{i=0}^r l_i.$$

In particular

$$H^r(X, \mathcal{O}_X(-r-1)),$$

is the free  $A$ -module with generator  $x_0^{-1} x_1^{-1} \dots x_r^{-1}$ . Hence (3).

To define a pairing, we declare

$$x_0^{l_0} x_1^{l_1} \dots x_r^{l_r},$$

to be the dual of

$$x_0^{m_0} x_1^{m_1} \dots x_r^{m_r} = x_0^{-1-l_0} x_1^{-1-l_1} \dots x_r^{-1-l_r}.$$

As  $m_i \geq 0$  if and only if  $l_i < 0$  it is straightforward to check that this gives a perfect pairing. Hence (4).

It remains to prove (2). If we localise the complex above with respect to  $x_r$ , we get a complex which computes  $\mathcal{F}|_{U_r}$ , which is zero in positive degree, as  $U_r$  is affine. Thus

$$H^i(X, \mathcal{F})_{x_r} = 0,$$

for  $i > 0$  so that every element of  $H^i(X, \mathcal{F})$  is annihilated by some power of  $x_r$ .

To finish the proof, we will show that multiplication by  $x_r$  induces an inclusion of cohomology. We proceed by induction on the dimension. Suppose that  $r > 1$  and let  $Y \simeq \mathbb{P}_A^{r-1}$  be the hyperplane  $x_r = 0$ . Then

$$\mathcal{I}_Y = \mathcal{O}_X(-Y) = \mathcal{O}_X(-1).$$

Thus there are short exact sequences

$$0 \longrightarrow \mathcal{O}_X(n-1) \longrightarrow \mathcal{O}_X(n) \longrightarrow \mathcal{O}_Y(n) \longrightarrow 0.$$

Now  $H^i(Y, \mathcal{O}_Y(n)) = 0$  for  $0 < i < r-1$  and the natural restriction map

$$H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)),$$

is surjective (every polynomial of degree  $n$  on  $Y$  is the restriction of a polynomial of degree  $n$  on  $X$ ). Thus

$$H^i(X, \mathcal{O}_X(n-1)) \simeq H^i(X, \mathcal{O}_X(n)),$$

for  $0 < i < r - 1$ , and even if  $i = r - 1$ , then we get an injective map. But this map is the one induced by multiplication by  $x_r$ .  $\square$

**Theorem 13.6** (Serre vanishing). *Let  $X$  be a projective variety over a noetherian ring and let  $\mathcal{O}_X(1)$  be a very ample line bundle on  $X$ . Let  $\mathcal{F}$  be a coherent sheaf.*

- (1)  $H^i(X, \mathcal{F})$  are finitely generated  $A$ -modules.
- (2) There is an integer  $n_0$  such that  $H^i(X, \mathcal{F}(n)) = 0$  for all  $n \geq n_0$ .

*Proof.* By assumption there is an immersion  $i: X \rightarrow \mathbb{P}_A^r$  such that  $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}_A^r}(1)$ . As  $X$  is projective, it is proper and so  $i$  is a closed immersion. If  $\mathcal{G} = i_*\mathcal{F}$  then

$$H^i(\mathbb{P}_A^r, \mathcal{G}) \simeq H^i(X, \mathcal{F}).$$

Replacing  $X$  by  $\mathbb{P}_A^r$  and  $\mathcal{F}$  by  $\mathcal{G}$  we may assume that  $X = \mathbb{P}_A^r$ .

If  $\mathcal{F} = \mathcal{O}_X(q)$  then the result is given by (13.5). Thus the result also holds if  $\mathcal{F}$  is a direct sum of invertible sheaves. The general case proceeds by descending induction on  $i$ . Now

$$H^i(X, \mathcal{F}) = 0,$$

if  $i > r$  (clear if we use Čech cohomology). On the other hand,  $\mathcal{F}$  is a quotient of a direct sum  $\mathcal{E}$  of invertible sheaves. Thus there is an exact sequence

$$0 \rightarrow \mathcal{R} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0,$$

where  $\mathcal{R}$  is coherent. Twisting by  $\mathcal{O}_X(n)$  we get

$$0 \rightarrow \mathcal{R}(n) \rightarrow \mathcal{E}(n) \rightarrow \mathcal{F}(n) \rightarrow 0.$$

Taking the long exact sequence of cohomology, for  $n$  large enough, we get isomorphisms

$$H^{i+1}(X, \mathcal{F}(n)) \simeq H^{i+1}(X, \mathcal{R}(n)),$$

and we are done by descending induction on  $i$ .  $\square$

**Corollary 13.7.** *Let  $A$  be a noetherian ring and let  $X$  be a proper scheme over  $A$ . Let  $\mathcal{L}$  be an invertible sheaf on  $X$ . TFAE*

- (1)  $\mathcal{L}$  is ample.
- (2) For every coherent sheaf  $\mathcal{F}$  on  $X$  there is an integer  $n_0$  such that

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0,$$

for  $n > n_0$ .

**Theorem 13.8** (Serre duality). *Let  $X$  be a smooth projective variety of dimension  $n$  over an algebraically closed field. Then there is an invertible sheaf  $\omega_X$  such that*

- (1)  $h^n(X, \omega_X) = 1$ .  
 (2) Given any other invertible sheaf  $\mathcal{L}$  there is a perfect pairing

$$H^i(X, \mathcal{L}) \times H^{n-i}(X, \omega_X \otimes \mathcal{L}^*) \longrightarrow H^n(X, \omega_X).$$

**Example 13.9.** Let  $X = \mathbb{P}_k^r$ . Then  $\omega_X = \mathcal{O}_X(-r-1)$  is a dualising sheaf. In fact, on any smooth projective variety, the dualising sheaf is constructed as the determinant of the cotangent bundle. To construct this as a sheaf, let  $i: X \rightarrow X \times X$  be the diagonal embedding. Let  $\mathcal{I}$  be the ideal sheaf of the diagonal and let

$$\Omega_X^1 = i^* \frac{\mathcal{I}}{\mathcal{I}^2}.$$

Then  $\Omega_X^1$  is a locally free sheaf of rank  $n$ , known as the sheaf of Kähler differentials. The determinant sheaf is then the dualising sheaf. This expresses a remarkable coincidence between the dualising sheaf, which is something defined in terms of sheaf cohomology and the determinant of the sheaf of Kähler differentials.