11. Divisors

Definition 11.1. *We say that a scheme X is **regular in codimension one** if every local ring of dimension one is regular.*

When talking about Weil divisors, we will only consider schemes which are

$(*)$ noetherian integral separated, regular in codimension one.

Definition 11.2. *Let X be a scheme satisfying $(*)$. A **prime divisor** Y on X is a closed integral subscheme of codimension one.*

*A **Weil divisor** D on X is an element of the free abelian group $\operatorname{Div} X$ generated by the prime divisors.*

Thus a Weil divisor is a formal linear combination $D = \sum_Y n_y Y$ of prime divisors, where all but finitely many $n_Y = 0$. We say that D is **effective** if $n_Y \geq 0$.

Definition 11.3. *Let X be a scheme satisfying $(*)$, and let Y be a prime divisor, with generic point η. Then $\mathcal{O}_{X,\eta}$ is a discrete valuation ring with quotient field K.*

*The **valuation** ν_Y **associated to** Y is the corresponding valuation.*

Noe that as X is separated, Y is determined by its valuation. If $f \in K$ and $\nu_Y(f) > 0$ then we say that f has a zero of order $\nu_Y(f)$; if $\nu_Y(f) < 0$ then we say that f has a pole of order $-\nu_Y(f)$.

Definition-Lemma 11.4. *Let X be a scheme satisfying $(*)$, and let $f \in K^*$.*

$$(f) = \sum_Y \nu_Y(f) Y \in \operatorname{Div} X.$$

Proof. We have to show that $\nu_Y(f) = 0$ for all but finitely many Y. Let U be the open subset where f is regular. Then the only poles of f are along $Z = X - U$. As Z is a proper closed subset and X is noetherian, Z contains only finitely many prime divisors.

Similarly the zeroes of f only occur outside the open subset V where $g = f^{-1}$ is regular. □

Any divisor D of the form (f) will be called **principal**.

Lemma 11.5. *Let X be a scheme satisfying $(*)$.*

The principal divisor are a subgroup of $\operatorname{Div} X$.

Proof. The map

$$K^* \longrightarrow \operatorname{Div} X,$$

is easily seen to be a group homomorphism. □

Definition 11.6. *Two Weil divisors D and D' are called* ***linearly equivalent****, denoted $D \sim D'$ if and only if the difference is principal. The group of Weil divisors modulo linear equivalence is called the* ***divisor Class group****, $\operatorname{Cl} X$.*

Proposition 11.7. *If k is a field then*

$$\operatorname{Cl}(\mathbb{P}^r_k) \simeq \mathbb{Z}.$$

Proof. Note that if Y is a prime divisor in $\mathbb{P}^n_k$ then the ideal of Y has height one, so that $I = \langle G \rangle$ and Y is defined by a single homogeneous polynomial G. The degree of G is called the degree of Y.

If $D = \sum n_y Y$ is a Weil divisor then define the degree $\deg D$ of D to be the sum

$$\sum n_Y \deg Y,$$

where $\deg Y$ is the degree of Y.

Note that the degree of any rational function is zero. Thus there is a well-defined group homomorphism

$$\deg\colon \operatorname{Cl} \mathbb{P}^r_k \longrightarrow \mathbb{Z},$$

and it suffices to prove that this map is an isomorphism. Let H be defined by X_0. Then H is a hyperplane and H has degree one. The divisor $D = nH$ has degree n and so the degree map is surjective. One the other hand, if $D = \sum n_i Y_i$ is effective, and Y_i is defined by G_i,

$$(\prod_i G^{n_i}/X_0^d) = D - dH,$$

where d is the degree of D. □

Example 11.8. *Let C be a smooth cubic curve in $\mathbb{P}^2_k$. Suppose that the line $Z = 0$ is a flex line to the cubic at the point $P_0 = [0 : 1 : 0]$. Then the equation of the cubic, in the affine piece $U_3 \simeq \mathbb{A}^2_k$, is of the form*

$$y^2 = x^3 + ax + b,$$

after completing the cube, for some a and $b \in k$.

Now two sets of three collinear points are linearly equivalent (since the equation of one line divided by another line is a rational function on the whole $\mathbb{P}^2_k$). In fact the whole class group is determined by only these linear equivalences.

Put differently, the rational points of C form an abelian group, where three points sum to zero if and only if they are collinear, and P_0 is declared to be the identity. The divisors of degree zero modulo linear equivalence are equal to this group.

In particular, an elliptic curve is very far from being isomorphic to $\mathbb{P}^1_k$.

Definition 11.9. *Given a ring A, let S be the multiplicative set of non-zero divisors of A. The localisation A_S of A at S is called the* ***total quotient ring*** *of A.*

Given a scheme X, let $\mathcal{K}$ the sheaf associated to the presheaf, which associates to every open subset $U \subset X$, the total quotient ring of $\Gamma(U, \mathcal{O}_X)$. $\mathcal{K}$ is called the ***sheaf of total quotient rings****.*

Definition 11.10. *A* ***Cartier divisor*** *on a scheme X is any global section of $\mathcal{K}^*/\mathcal{O}^*_X$.*

In other words, a Cartier divisor is specified by an open cover U_i, a collection of rational functions f_i, such that f_i/f_j is a nowhere zero regular function.

A Cartier divisor is called **principal** if it is in the image of $\Gamma(X, \mathcal{K}^*)$. Two Cartier divisors D and D' are called **linearly equivalent**, denoted $D \sim D'$ if and only if the difference is principal.

Definition 11.11. *Let X be a scheme satisfying $(*)$. Then every Cartier divisors determines a Weil divisor.*

Informally a Cartier divisor is simply a Weil divisor defined locally by one equation. If every Weil divisor is Cartier then we say that X is factorial. This is equivalent to requiring that every local ring is a UFD; for example every smooth variety is factorial.

Example 11.12. *The quadric cone Q, given by $xy - z^2 = 0$ in $\mathbb{A}^3_k$ is not factorial. The line l, given by $x = z = 0$, is a Weil divisor which is not Cartier (one needs to check that the ideal $\langle x, z \rangle$ inside $\mathcal{O}_{X,0}$ is not principal). The hyperplane $x = 0$ cuts out the double line $2l$.*

Definition-Lemma 11.13. *Let X be a scheme.*

The set of invertible sheaves forms an abelian group $\operatorname{Pic}(X)$*, where multiplication corresponds to tensor product and the inverse to the dual.*

Proof. It is clear that tensor product is commutative and associative and that $\mathcal{O}_X$ plays the role of the identity. But if $\mathcal{M} = \operatorname{Hom}(\mathcal{L}, \mathcal{O}_X)$ then

$$\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathcal{L} \simeq \operatorname{Hom}(\mathcal{L}, \mathcal{L}) \simeq \mathcal{O}_X. \qquad \square$$

Definition 11.14. *Let D be a Cartier divisor, represented by $\{(U_i, f_i)\}$. Define a subsheaf $\mathcal{O}_X(D) \subset \mathcal{K}$ by taking the subsheaf generated by f_i^{-1} over the open set U_i.*

Proposition 11.15. *Let X be a scheme.*

(1) *The association* $D \longrightarrow \mathcal{O}_X(D)$ *defines a correspondence between Cartier divisors and invertible subsheaves of* $\mathcal{K}$.
(2) *If* $\mathcal{O}_X(D_1 - D_2) \simeq \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}$.
(3) *Two Cartier divisors* D_1 *and* D_2 *are linearly equivalent if and only if* $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$.