SEVENTH HOMEWORK, DUE THURSDAY APRIL 10TH

1. Let \( x \in X \) be a point of a scheme, with residue field \( k \).
   (i) Show that the Zariski tangent space, \( T_x X \) as defined in (8.1) is indeed a \( k \)-vector space.
   (ii) Show that if \( m \subset O_{X,x} \) is the maximal ideal, then there is a natural isomorphism of \( k \)-vector spaces,
   \[ T_x X \simeq \left( \frac{m}{m^2} \right)^*. \]
   (iii) What is the dimension of the Zariski tangent spaces to the zero dimensional schemes of length at most four over an algebraically closed field?
   (iv) Show that there are schemes of finite type over an algebraically closed field, where the dimension of the Zariski tangent space at any point is always greater than the dimension of the scheme.

2. An **affine bundle of dimension** \( n \) is a morphism \( \pi: X \longrightarrow Y \) of schemes such that there is a open cover \( \{U_i\} \) of \( Y \) isomorphisms
   \[ \phi_i: \pi^{-1}(U_i) \longrightarrow \mathbb{A}^n_{U_i}, \]
   for each \( U_i \), where moreover the transition functions
   \[ g_{ij} = \phi_j \circ \phi_i^{-1}: \mathbb{A}^n_{U_{ij}} \longrightarrow \mathbb{A}^n_{U_{ij}}, \]
   are affine linear (same as linear automorphism, but where we also allow translation, \( x \longrightarrow Ax + b \)). If \( \pi \) has a section, so that the fibres are vector spaces and (we can arrange that) the transition functions are linear (that is they fix the section), then we say that \( \pi \) is a vector bundle of rank \( n \).
   (i) Let \( \mathcal{H}^l \) be the punctual Hilbert scheme of length \( l \) zero dimensional schemes in \( \mathbb{A}^n_k \) supported at the origin. Let \( \mathcal{C}^l \) be the locally closed subset (with the reduced induced structure) of curvilinear schemes.
   Show that \( \mathcal{C}^2 \) is a copy of \( \mathbb{P}^{n-1}_k \).
   (ii) Show that there is a natural morphism
   \[ \pi_l: \mathcal{C}^l \longrightarrow \mathcal{C}^{l-1}, \]
   which realises \( \mathcal{C}^l \) as an affine bundle of dimension \( n - 1 \) over \( \mathcal{C}^{l-1} \), for any \( l \geq 3 \).
   (iii) Show that \( \pi_2 \) has a section, and that with this section \( \pi_2 \) becomes a vector bundle of rank \( n - 1 \) over \( \mathbb{P}^{n-1} \).
(iv) Now suppose that $n = 2$ and let $\pi = \pi_3 \circ \pi_4 : X = \mathbb{C}^4 \to \mathbb{C}^2 = \mathbb{P}^1_k$. Show that $\pi$ is an affine bundle of dimension two with a section.