SIXTH HOMEWORK, DUE THURSDAY APRIL 3RD

1. Let X be a smooth variety of dimension n over an algebraically closed field k of characteristic zero. What is the dimension of the space of curvilinear schemes of length l?

2. Let k be an algebraically closed field of characteristic zero. Fix integers p, q and n, and let $\mathfrak{m} \subset k[x_1, x_2, \dots, x_n]$ be the maximal ideal of the origin in $\mathbb{A}^n_k$. Let $A = k[x_1, x_2, \dots, x_n]/I$ where $I \triangleleft k[x_1, x_2, \dots, x_n]$ is any ideal such that

$$\mathfrak{m}^{p+1} \subset I \subset \mathfrak{m}^p,$$

and the quotient $W = I/\mathfrak{m}^{p+1} \subset \mathfrak{m}^p/\mathfrak{m}^{p+1} = V$ is a vector subspace of dimension q.

(i) What is the length of $z = \operatorname{Spec} A$?

(ii) Show that there is a correspondence between ideals I and subvector spaces $W \subset V$.

(iii) What is the dimension of the space of all such zero dimensional subschemes $z \subset \mathbb{A}^n_k$ supported at the origin? (*Hint: Use the Grassmannian. You may assume the naive dimension count is correct.*)

3. Comparing your answers to (1) and (2) show that the space of zero dimensional schemes in $\mathbb{A}^n$ is not the closure of the locus of curvilinear schemes (*Hint: It is for $n = 2$ (don't prove this; it is hard!) but it is not for $n = 3$*).