MODEL ANSWERS TO THE FOURTH HOMEWORK

1. (i) Here is a pretty sneaky way to solve this problem. First note that this problem is étale local about any pair of points p and q. But any smooth curve is étale locally equivalent to $\mathbb{P}^1$. So we may assume that $C = \mathbb{P}^1$. In this case divisors of degree two can be identified with polynomials of degree two, modulo constants, that is $\mathbb{P}^2$, which is smoooth.

Alternatively, working locally, we may assume that $C = \operatorname{Spec} \mathbb{C}[t]$ is affine. On $C \times C$ we have coordinates x and y, so that $C \times C = \operatorname{Spec} \mathbb{C}[x, y]$. Then $C_2 = \operatorname{Spec} \mathbb{C}[x, y]^{\mathbb{Z}_2}$. Now the action of $\mathbb{Z}_2$ is the standard one, given by swapping x and y. The ring of invariants is

$$\mathbb{C}[x, y]^{\mathbb{Z}_2} = \mathbb{C}[x + y, xy] = \mathbb{C}[u, v],$$

by a classical result, whose proof goes back to Newton. But then C_2 is smooth.

(ii) Clear.

(iii) Consider the linear system $|K_C| \subset C_2$. The Abel-Jacobi map collapses this to a point, since by definition the fibres of the Abel-Jacobi map are linear systems. By Riemann-Roch, $|K_C|$ is a g^1_2 and so defines a copy of $\mathbb{P}^1 \subset C_2$.

(iv) $\overline{\mathrm{NE}}(C_2) \subset \mathbb{R}^2$. Thus there are two rays, of which one is given by the g^1_2. The first thing to do is compute the intersection pairing between the basis elements x and δ. We will use push-pull, which we may state in the form

$$\alpha \cdot f^*\beta = f_*(\alpha \cdot f^*\beta) = f_*\alpha \cdot \beta,$$

where $f \colon X \longrightarrow Y$ is a finite morphism of degree d, α is a cycle of dimension k on X and β is a cycle of dimension $n - k$ on Y.

We apply this to the natural map $f \colon C \times C \longrightarrow C_2$ of degree 2. First note that $f^*\delta = \Delta \subset C \times C$, where Δ is the diagonal. Now $\Delta \simeq C$, and

$$2 = K_C = (K_S + \Delta) \cdot \Delta = 2K_C + \Delta \cdot \Delta = 4 + \Delta^2.$$

Thus $\Delta^2 = -2$. But then

$$\delta^2 = f^*\delta \cdot \Delta = 2\Delta^2 = -4.$$

In particular δ generates the other extremal ray. Now note that

$$f^*x = X = X_1 + X_2 = [\{p\} \times C] + [C \times \{p\}],$$

where $p \in C$ is any point.

$$x^2 = f^* x \cdot X_1 = 1.$$

Finally,

$$x \cdot \delta = X_1 \cdot f^* \delta = 2 X_1 \cdot \Delta = 2.$$

We want to calculate the class $\gamma = ax + b\delta$ of the g^1_2. We have

$$1 = \gamma \cdot x = (ax + b\delta) \cdot x = a + 2b.$$

How about $\gamma \cdot \delta$? This counts the number r of ramification points of the g^1_2. By Riemann-Hurwitz,

$$2 = 2g - 2 = -2 \cdot 2 + r = r,$$

so that $r = 6$. Thus

$$6 = \gamma \cdot \delta = (ax + b\delta) \cdot \delta = 2a - 4b.$$

Thus $a = 2$, $b = -1/2$ and $\gamma = 2x - \delta/2$. To check this calculation, note that as γ represents a copy of $\mathbb{P}^1$ contracted to a smooth point, in fact

$$-1 = \gamma^2 = (2x - \delta/2)^2 = 4x^2 - 2x \cdot \delta - \delta^2/4 = 4 - 4 - 4/4 = -1.$$

Thus $\overline{\mathrm{NE}}(C_2)$ is the two dimensional cone spanned by δ and $4x - \delta$.
(v) It is classical that the image of x represents the class θ of the Θ divisor. The Θ divisor is ample. $\pi^*\theta = x + c\gamma$, where c is determined by the constraint that

$$0 = \pi^*\theta \cdot \gamma = (x + c\gamma) \cdot \gamma = 1 - c.$$

Thus $c = 1$. The Seshadri contstant is the largest d such that

$$\pi^*\theta - d\gamma$$

is nef. Since this dots γ positively, we want to find d such that

$$(\pi^*\theta - d\gamma) \cdot \delta = 0,$$

that is

$$2 - 6(d - 1) = (x - (d - 1)\gamma) \cdot \delta = 0.$$

Thus $d = 4/3$.
2. Note that the kernel of a q-linear map is a linear subspace. Let K_i be the kernel of the q^i-linear map f^i. Then

$$0 \subset V_1 \subset V_2 \subset V_3 \subset \dots,$$

is an ascending chain of linear subspaces of V. This chain must stabilise; suppose that it stabilises at V_n. Then V_n is a linear subspace of V, consisting of those elements of V for which some power of f is zero. Certainly $f|_{V_n}$ is nilpotent, and $f(V_n) \subset V_n$. Now note that the image of a q-linear map is a linear subspace (since we are working over an algebraically closed field; in fact we only need that the groundfield is separably closed). Let W_i be the image of f^i. Then

$$V \supset W_1 \supset W_2 \supset W_3 \supset \dots,$$

is a descending chain of linear subspaces of V. This chain must stabilise; suppose that it stabilises at V_s. Then V_s consists of those elements of V which are in the image of all powers of f. It follows that $f(V_s) = V_s$. Arguing as in the case of linear maps, $f|_{V_s}$ has an eigenvector, so that V_s has a basis of eigenvectors, that is

$$f(v_i) = \lambda_i v_i,$$

for some scalar $\lambda_i \neq 0$. Pick a_i so that $a_i^{q-1} = \lambda_i^{-1}$. Then

$$f(w_i) = w_i,$$

where $w_i = a_i v_i$ is a basis of eigenvectors, with eigenvalue one. Finally suppose that $\alpha \in V$. Let $\alpha_k = f^k(\alpha)$. Then we may choose k sufficiently large, so that $\alpha_k \in V_s$. Suppose that $f^k(\beta) = \alpha_k$, where $\beta \in V_s$, and let $\gamma = \alpha - \beta$. Then

$$f^k(\gamma) = f^k(\alpha) - f^k\beta) = 0.$$

Thus

$$V = V_s + V_n.$$

On the other hand, it is clear that the intersection is trivial.

3. A cylic cover $f\colon Y \longrightarrow X$ of degree p is given locally by an equation of the form $y^p - y - a = 0$, where $a \in \mathcal{O}_U$ is some regular function on an open affine U of X. If $U = \operatorname{Spec} A$ then $V = f^{-1}(U) = \operatorname{Spec} B$, where

$$B = \frac{A[y]}{\langle y^p - y - a_i\rangle}.$$

Given f, take a cover $\{U_i\}$ and functions $a_i \in \mathcal{O}_{U_i}$ such that Y is given locally by $y_i^p - y_i - a_i = 0$. We may assume that the overlaps $U_{ij} = U_i \cap U_j = \operatorname{Spec} A_{ij} = A$ are affine, and in this case we have an isomorphism of A-algebras,

$$\frac{A[y_i]}{\langle y_i^p - y_i - a_i\rangle} \simeq \frac{A[y_j]}{\langle y_j^p - y_j - a_j\rangle},$$

where we assume that y_i and y_j correspond under this isomorphism. Consider the 1-cycle,

$$\alpha = (y_{ij} = y_i - y_j) \in H^1(X, \mathcal{O}_X).$$

Then

$$F^*\alpha = (y_{ij}^p = y_i^p - y_j^p = y_i - y_j - (a_i - a_j)) = \alpha + \delta(y_i).$$

Thus $F^*\alpha = \alpha$, and it is easy to reverse this argument.

4. (i) By Keel's Theorem, it suffices to show that $C|_{\mathbb{E}(D)}$ is semiample. Suppose that B is a component of $\mathbb{E}(D)$. Then $C \cdot B = 0$. The space P of line bundles of degree zero on B is a projective variety. The line bundle $L = \mathcal{O}_B(C)$ is an element of $P(\mathbb{F}_q)$, which is a finite group G. But then L must have finite order, so that $C|_B$ is semiample.

(ii) The space P of line bundles of degree zero on C is a projective variety. The line bundle $L = \mathcal{O}_C(C)$ is an element of $P(\mathbb{F}_q)$, which is a finite group G. But then L must have finite order, so that $C|_C$ is torsion.

(iii) By induction on k. There is a short exact sequence

$$0 \longrightarrow \mathcal{O}_S((k-1)C) \longrightarrow \mathcal{O}_S(kC) \longrightarrow \mathcal{O}_C(kC) \longrightarrow 0.$$

If $k = 1$, then we get an isomorphism

$$H^1(S, \mathcal{O}_S(C)) \simeq H^1(S, \mathcal{O}_S).$$

By assumption $h^0(C, \mathcal{O}_C(kC)) = 0$, for $0 < k < m$. But then by Riemann-Roch $h^1(C, \mathcal{O}_C(kC)) = 0$, in the same range. It follows that

$$H^1(S, \mathcal{O}_S((k-1)C)) \simeq H^1(S, \mathcal{O}_S(kC)),$$

for $0 < k < m$. But then

$$H^1(S, \mathcal{O}_S(kC)) \simeq H^1(S, \mathcal{O}_S),$$

in the same range.

(iv) By (iii) it suffices to evaporate

$$H^1(S, \mathcal{O}_S),$$

which we did in class. Suppose that $f \colon T \longrightarrow S$ evaporates

$$H^1(S, \mathcal{O}_S((m-1)C)) \simeq H^1(S, \mathcal{O}_S).$$

There is a commutative diagram

$$\begin{array}{ccccc} H^0(S, \mathcal{O}_S(mC)) & \longrightarrow & H^0(C, \mathcal{O}_C) & \longrightarrow & H^1(S, \mathcal{O}_S((m-1)C)) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(T, \mathcal{O}_T(mD)) & \longrightarrow & H^0(D, \mathcal{O}_D) & \longrightarrow & H^1(T, \mathcal{O}_T((m-1)D)). \end{array}$$

Let $\sigma \in H^0(C, \mathcal{O}_C)$ be any non-vanishing section. Suppose that its image in $H^1(S, \mathcal{O}_S((m-1)C))$ is δ. If δ is not zero, then we cannot lift σ to an element of $H^0(S, \mathcal{O}_S(mC))$. Let σ' be the image of σ in $H^0(D, \mathcal{O}_D)$. Then σ' is also a non-vanishing section. Let δ' be the image of σ' in $H^1(T, \mathcal{O}_T((m-1)D))$. Then δ' is also the image of δ. But then by our choice of f, $\delta' = 0$.