2. The canonical divisor

In this section we will introduce one of the most important invariants in the birational classification of varieties.

Definition 2.1. *Let X be a normal quasi-projective variety of dimension n.*

*A **Weil divisor** is a formal linear combination of codimension one subvarieties. The set of all divisors with integer coefficients forms a group, which is nothing but the free abelian group with generators the irreducible and reduced divisors, aka the prime divisors. A $\mathbb{Q}$-divisor is a divisor with rational coefficients and an $\mathbb{R}$-divisor is a divisor with real coefficients.*

Divisors on smooth curves are very easy to understand. A $D = \sum_{p\in C} n_p p$ on a curve is nothing more than a formal sum of points, where all but finitely many of the coefficients n_p are zero.

Definition 2.2. *Let $D = \sum_{p\in C} n_p p$ be a divisor on a smooth curve C. The **degree** of D is the sum*

$$\sum n_p.$$

There are two very natural ways to construct integral divisors:

Definition 2.3. *Let X be a normal quasi-projective variety and let $f \in K(X)$ be a rational function. We associate to f the divisor of zeroes minus the divisor of poles:*

$$\begin{aligned}(f) &= (f)_0 - (f)_\infty \\ &= \sum_{V\subset X} \operatorname{mult}_V f,\end{aligned}$$

where the sum ranges over every irreducible subvariety $V \subset X$ of codimension one.

*We say that two divisors D_1 and D_2 are **linearly equivalent**, denoted $D_1 \sim D_2$ if*

$$D_1 = D_2 + (f),$$

where f is a rational function. The group of integral Weil divisors (ie those Weil divisors with integer coefficients) modulo linear equivalence is denoted $Z_{n-1}(X)$.

Example 2.4. *Let $X = \mathbb{P}^n$. Then the group of integral Weil divisors modulo linear equivalence is equal to $\mathbb{Z}$. Indeed define a map*

$$\pi\colon \mathbb{Z} \longrightarrow \mathbb{Z}_{n-1}(\mathbb{P}^n),$$

by sending

$$d \longrightarrow dH,$$

where H is the hyperplane defined by X_0. We first show that ρ is surjective. Suppose that $V \subset \mathbb{P}^n$ is a divisor. Then V is a hypersurface and it is defined by a homogeneous polynomial F of degree d. Then

$$f = \frac{F}{X_0^d},$$

is a rational function so that

$$V \sim dH,$$

where H is the hyperplane defined by X_0. Thus ρ is surjective.

Now suppose that $\rho(d) = 0$. Note that

$$0 = (dH) \cdot L = d,$$

where L is a line. Thus $d = 0$ and so ρ is injective.

The following easy result will be used so often it is useful to record it as a:

Lemma 2.5. *Let X be a normal variety and let U be an open subset whose complement has codimension at least two.*

Then every Weil divisor on X is determined by its restriction to U.

Proof. Suppose that $B = \sum a_i B_i$ is a Weil divisor on U. Let D_i be the closure of B_i. Then $D = \sum a_i D_i$ is a Weil divisor on X whose restriction to U is equal to B. Uniqueness is equally clear. □

Definition-Lemma 2.6. *Let X be a normal variety. We are going to associate a divisor to X. Note that the singular locus of X has codimension at least two. Thus by (2.5) we may assume that X is smooth. Let ω be a rational n-form. Then the zeroes minus the poles of ω determine a divisor, K_X, called the* ***canonical divisor****. The canonical divisor is well-defined up to linear equivalence.*

Proof. Suppose that η is any other rational n-form, with zeroes minus poles K'_X. The key point is that the ratio $f = \omega/\eta$ is a rational function. Thus

$$K_X = K'_X + (f).$$ □

There are two reasons that the canonical divisor is so useful as an invariant. One is that it is relatively easy to compute:

Example 2.7. *We will show that $K_{\mathbb{P}^n} = -(n+1)H$. To specify a rational n-form, it suffices to start with a rational n-form on an open* *affine subset, and compute what it looks like on the other open charts. Consider*

$$\omega = \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \cdots \wedge \frac{dx_n}{x_n},$$

on the affine chart U_0, given by $X_0 \neq 0$, with coordinates

$$x_i = \frac{X_i}{X_0}.$$

Then this has a pole along every hyperplane $x_i = 0$, $i > 0$. Thus the hyperplane H_i given by $X_i = 0$ occurs with multiplicity -1 for the corresponding canonical divisor. Since U_0 does not have codimension two, it remains to check that the multiplicity of H_0 of is also -1. Assuming this we have

$$K_{\mathbb{P}^n} = -(H_0 + H_1 + \cdots + H_n) \sim -(n+1)H.$$

Typically any formula for computing the canonical divisor comes with a fancy name:

Theorem 2.8 (Adjunction formula)**.** *Let X be a smooth variety and let S be a smooth divisor.*

Then

$$(K_X + S)|_S = K_S.$$

Proof. The easiest way to prove this is to realise the canonical divisor as the first chern class of the cotangent bundle T^*_X. There is a short exact sequence

$$0 \longrightarrow T_S \longrightarrow T_X \longrightarrow N_{S/X} \longrightarrow 0.$$

Since the first chern class is additive on exact sequences, we have

$$-K_X = c_1(T_X) = c_1(T_S) + c_1(N_{S/X}) = -K_S + c_1(N_{S/X}).$$

It remains to determine the normal bundle $N_{S/X}$.

Claim 2.9. $N_{S/X} = \mathcal{O}_S(S)$.

Proof of (2.9). Suppose that U_α is an open cover of X and that $S \cap U_\alpha$ is defined by f_α. On overlaps, we have

$$f_\beta = u_{\alpha\beta} f_\alpha,$$

where $u_{\alpha\beta} \in \mathcal{O}_{U_{\alpha\beta}}$ is a unit. Thus the ideal sheaf

$$\mathcal{I}_{S/X} = \mathcal{O}_S(-S),$$

is the line bundle with transition functions $u_{\alpha\beta}$.

Consider the differential form df_α. This is a section of $T^*_X|_{U_\alpha}$, and by restriction we get a section of the conormal bundle $N^*_{S/X}$. We have

$$df_\beta = d(u_{\alpha\beta} f_\alpha) = du_{\alpha\beta} f_\alpha + u_{\alpha\beta} df_\alpha.$$

Now the first term vanishes on S, due to the factor f_α. Thus $N^*_{S/X}$ is a line bundle with the same transition functions as $\mathcal{O}_S(-S)$. Thus the two line bundles $N^*_{S/X}$ and $\mathcal{O}_S(-S)$ are isomorphic. Dualising establishes the claim. $\square$

Thus

$$c_1(N_{S/X}) = S|_S,$$

and rearranging we get the adjunction formula. $\square$

One interesting feature of the adjunction formula is that it suggests that instead of working with canonical divisors we ought to work with canonical divisors plus other divisors:

Definition 2.10. *Let X be a normal variety. We say that a divisor D is **Cartier** if D is locally defined by a single equation.*

The key point of Cartier divisors is that given a morphism $\pi\colon Y \longrightarrow X$ whose image does not lie in D, then we can pullback a Cartier divisor to Y. Indeed, just pull back local defining equations. In particular suppose that we are given a curve $C \subset X$ or more generally a morphism $f\colon C \longrightarrow X$, whose image does not lie in D. Then we can define the intersection number of D and C,

$$D \underset{f}{\cdot} C = \deg f^*D.$$

More generally one can intersect a Cartier divisor with any subvariety and get a Cartier divisor on the subvariety, again provided the subvariety is not contained in the Cartier divisor. Unfortunately using this, it is all too easy to give examples of integral Weil divisors which are not Cartier:

Example 2.11. *Let $X \subset \mathbb{P}^3$ be the quadric cone, which is given locally as $X_0 = (xy - z^2) \subset \mathbb{A}^3$. Then the line $L = (x = z = 0) \subset \mathbb{A}^3$ is a Weil divisor which is not Cartier. Indeed, let us compute the self-intersection $L^2 = L \cdot L$. First note that L is linearly equivalent to the line $M = (y = z = 0) \subset \mathbb{A}^3$. Thus*

$$L^2 = M \cdot L.$$

Now note that the hyperplane $H = (Y = 0) \subset \mathbb{P}^3$ cuts out twice the line $2M$. Indeed the hyperplane is everywhere tangent to X along M. If L were Cartier then

$$2(M \cdot L) = (2M) \cdot L = H \cdot L = 1,$$

a contradiction.

Definition 2.12. *Let X be a normal variety, and let $D \subset X$ be a $\mathbb{Q}$-divisor. We say that D is $\mathbb{Q}$-Cartier if nD is Cartier for some integer n.*

We say that a normal variety is $\mathbb{Q}$-factorial if every Weil divisor is $\mathbb{Q}$-Cartier.

In the example above, $2L$ is Cartier. In fact it is defined by the equation $x = 0$ on the quadric. In fact the quadric cone is $\mathbb{Q}$-factorial. Indeed if D is any integral Weil divisor then $2D$ is always Cartier.

As an aside, one can always define the intersection number of a curve C with a Cartier divisor. The clumsy way to do this is to proceed as above, and deform the divisor to a linearly equivalent divisor, which does not contain the curve. A more sophisticated approach is as follows. If the image of the curve lies in the divisor, then instead of pulling the divisor back, pullback the associated line bundle and take the degree of that

$$D \underset{f}{\cdot} C = \deg f^*\mathcal{O}_X(D).$$

Definition 2.13. *We say that (X, Δ) is a* ***log pair*** *if X is a normal variety and $\Delta \geq 0$ is a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say that $\Delta = \sum a_i \Delta$ is a* ***boundary*** *if $a_i \in [0, 1]$.*

Part of the reason for the justification for this definition is given by:

Theorem 2.14 (Riemann-Hurwitz formula)**.** *Let $f : Y \longrightarrow X$ be a finite morphism between normal quasi-projective varieties. Let $\Delta = \sum a_i \Delta_i$ contain the support of the branch locus (to achieve this, one might have to throw in components with coefficient zero).*

Then we may write

$$K_Y + \Gamma = f^*(K_X + \Delta),$$

where $\Gamma = \sum_i b_i \Gamma_i$, the sum runs over the prime components of the ramification divisor and if $f(\Gamma_i) = \Delta_j$ then

$$b_i = r_i a_i - (r_i - 1),$$

where r_i is a positive integer, known as the ***ramification index*** *of f at Γ_i.*

Proof. Since we are asserting an equality of Weil divisors, it suffices to check that the coefficients are correct. If the dimension of Y is greater than one, to check this we can restrict to a general hyperplane S in X. Throwing a subset of codimension at least two, we may assume that X and Y are smooth.

Suppose that T is the inverse image of S. Since X and Y are smooth and S and T are general, S and T are smooth (Bertini's Theorem). By adjunction we have

$$(K_X + S)|_S = K_S \qquad \text{and} \qquad (K_Y + T)|_T - K_T.$$

Repeatedly replacing Y by a hyperplane and replacing X by the inverse image of a hyperplane, we may assume that X and Y are smooth curves, or Riemman surfaces.

Let q be a point in Y with image p in X. Then the result is local about p and q. But any map between two Riemann surfaces is locally given as

$$z \longrightarrow z^k = t,$$

for some positive integer k, which is the ramification index, where $z = 0$ defines q and $t = 0$ defines p. As

$$f^*t = z^k,$$

it follows that

$$f^*p = kq.$$

On the other hand,

$$f^*(dt) = dz^k = kz^{k-1}dz.$$

Thus,

$$f^*K_X + (k-1)q = K_Y.$$

locally about p and q. Putting all of this together gives the result. $\square$

There are some very interesting special cases of (2.14). For example suppose that the ramification index r only depends on the branch divisor (for example if the map f is Galois, so that $X = Y/G$, for some finite group G; in this case the ramification index is simply the order of the stabiliser). In this case if

$$\Delta = \sum_i \frac{r_i - 1}{r_i}\Delta_i,$$

then

$$K_Y = f^*(K_X + \Delta).$$

For this very reason, coefficients of the form $(r-1)/r$ play a central role in log geometry. A very special case of this is when the map f is unramified (aka étale). In this case

$$K_Y = f^*K_X.$$

Note that (unfortunately) it is often the case that Δ is a boundary and yet Γ is not. The problem is that some of the coefficients of Γ might be negative. In fact it is necessary and sufficient that the coefficient of the branch divisor is at least the coefficient $(r-1)/r$, where r is the largest ramification index lying over this divisor. Again a very special but interesting case of this is when the coefficients of Δ are all one. In this case the coefficients of Γ are also all one.

We end this section with some of the most fundamental properties of the canonical divisor:

Theorem 2.15 (Serre Duality). *Let L be a line bundle on a smooth (or more generally Cohen-Macaulay) variety X of dimension n. Then there are natural isomorphisms*

$$H^i(X, L)^* \simeq H^{n-i}(X, L^*(K_X)).$$

One normally states this result by saying that there is an isomorphism $\omega_X \simeq \mathcal{O}_X(K_X)$, where ω_X is the dualising sheaf. The Cohen-Macaulay condition is just the condition that there is a dualising sheaf (ie a sheaf which makes Serre duality work). Thus (2.15) is very strong; whenever Serre duality is true, the duality is given by the canonical divisor.

Theorem 2.16 (Riemann-Roch). *Let C be a smooth curve of genus g and let D be an integral divisor on X of degree d. Then*

$$h^0(C, \mathcal{O}_C(D)) - h^0(C, \mathcal{O}_C(K_C - D)) = d - g + 1.$$

Proof. By Serre duality, it suffices to prove that

$$\chi(\mathcal{O}_C(D)) = h^0(C, \mathcal{O}_C(D)) - h^1(C, \mathcal{O}_C(D)) = d - g + 1.$$

Let E be any divisor of degree e and let p be any point. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_C(-p) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

Here $\mathcal{O}_p$ is a skyscraper sheaf, supported at the single point p. Twisting by the divisor $E + p$ we have

$$0 \longrightarrow \mathcal{O}_C(E) \longrightarrow \mathcal{O}_C(E+p) \longrightarrow \mathcal{O}_p(E) \longrightarrow 0.$$

Taking the long exact sequence associated to the short exact sequence and using the additivity of the Euler characteristic we have:

$$\chi(\mathcal{O}_C(E+p)) = \chi(\mathcal{O}_C(E)) + \chi(\mathcal{O}_p) = \chi(\mathcal{O}_C(E)) + 1,$$

where we used the fact that $h^1(C, \mathcal{O}_p) = 0$. Since the formula on the RHS of Riemann-Roch is linear it follows that the Riemann-Roch formula holds for E iff the Riemann-Roch formula holds for $E + p$.

In particular it suffices to prove that Riemann-Roch holds for $D + kp$, where k is as large as we please. But if k is sufficiently large then

$|D+kp|$ is non-empty, so that there is an effective divisor $D' \sim D+kp$. Replacing $D+kp$ by $D' \geq 0$ we may thus assume that $D \geq 0$.

Now we proceed by induction on d, the degree of D. If $d > 0$ then $D = E + p$ where $E \geq 0$ has degree $d-1$. By what we have already proved, we may therefore assume that $d = 0$. But then $D = 0$ so that

$$\chi(\mathcal{O}_C(D)) = h^0(C, \mathcal{O}_C) - h^1(C, \mathcal{O}_C) = 1 - g,$$

as required. $\square$