# Polynomial equations

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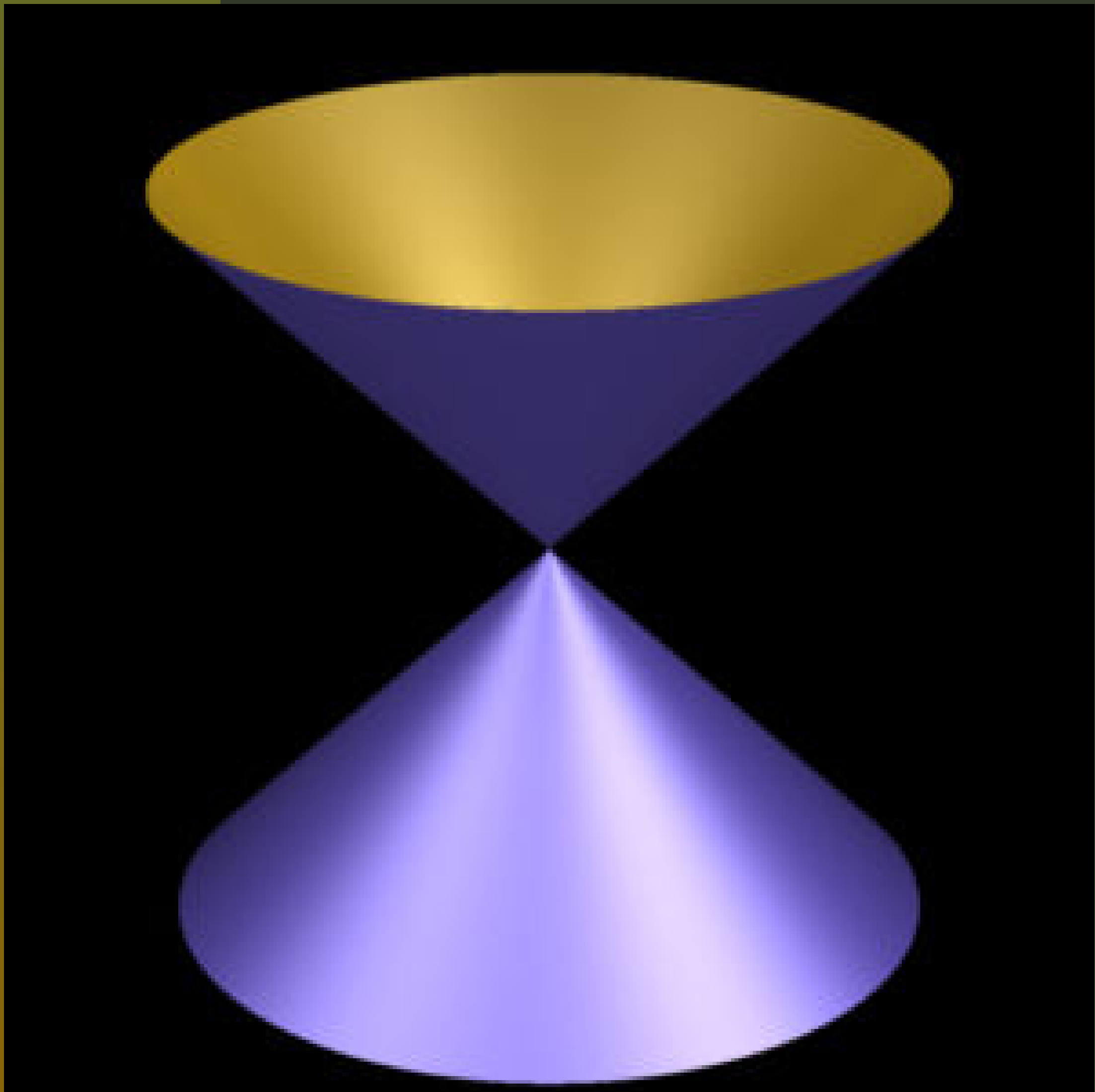
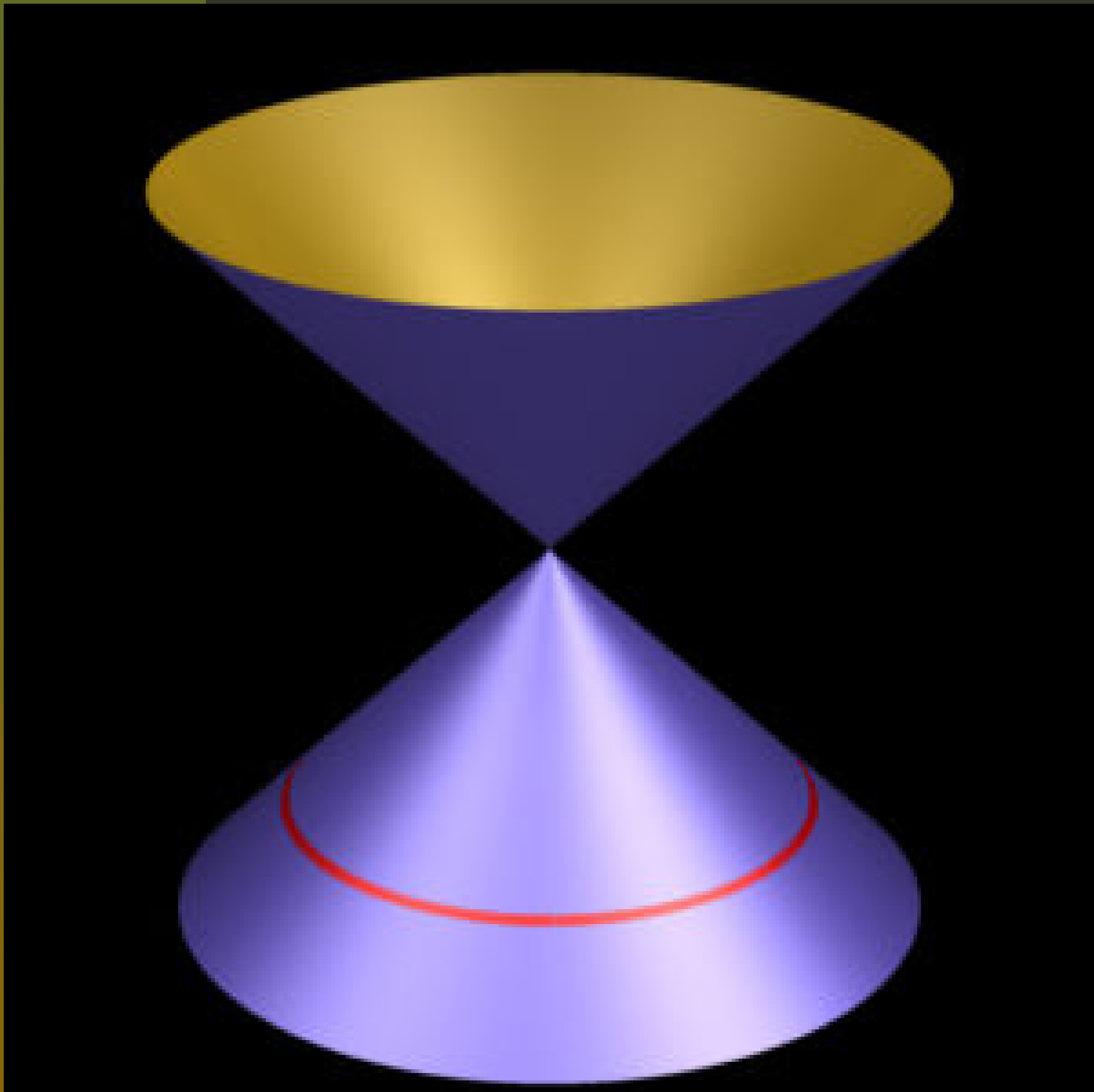
MIT

# Classical geometry: conic sections

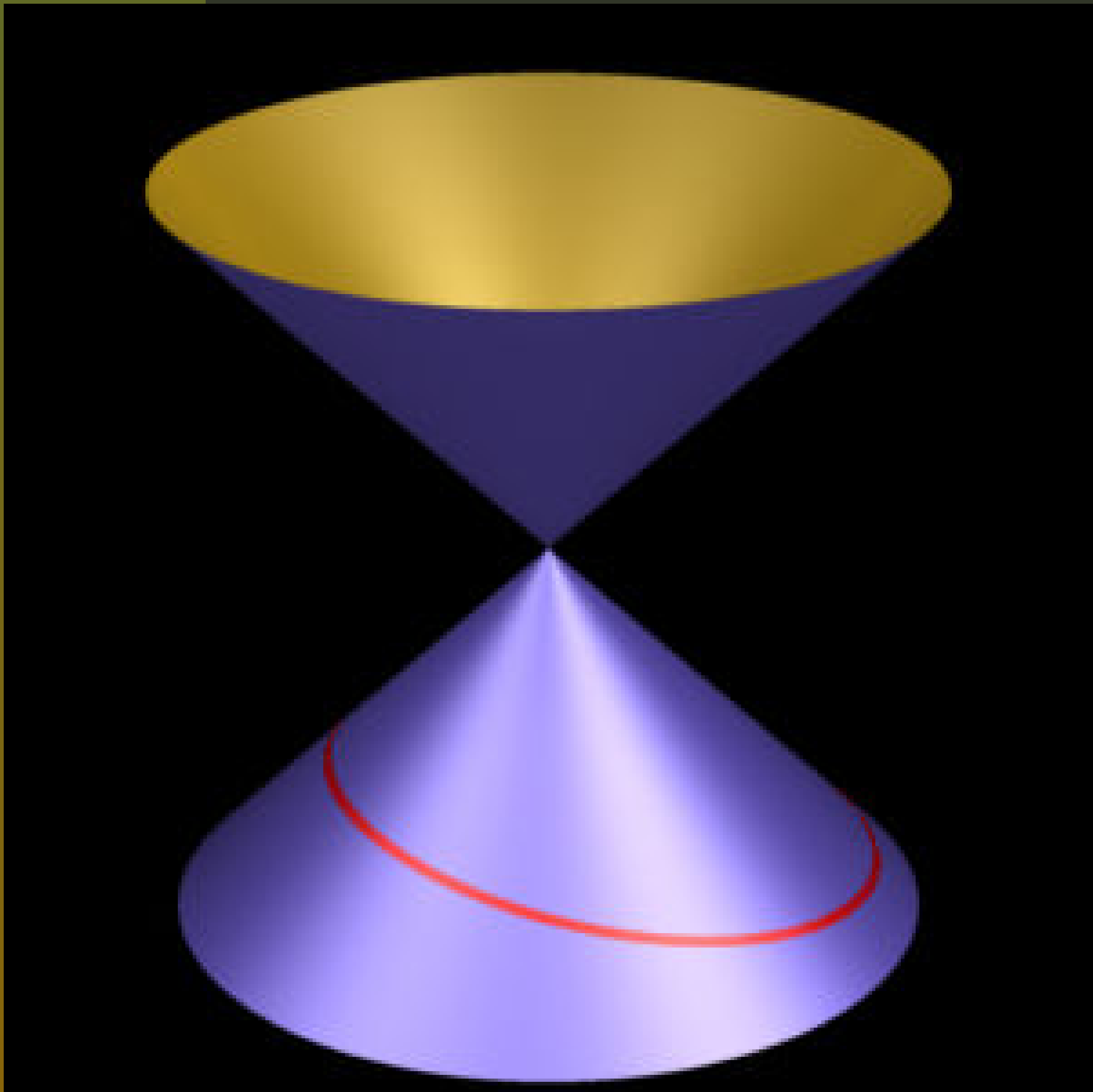
Menaechmus studied conic sections in 3rd century BC:



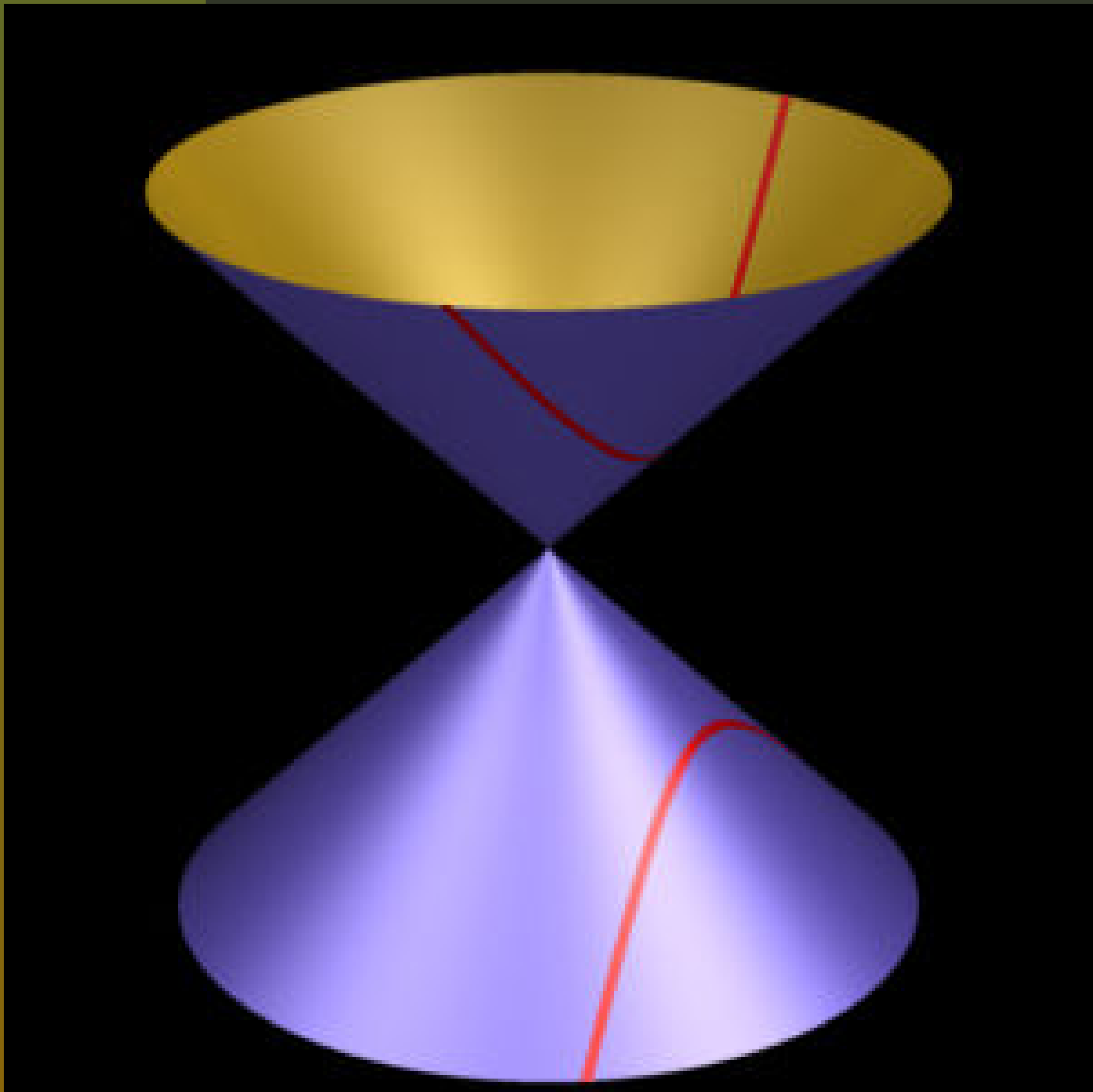
# Conic sections: circle



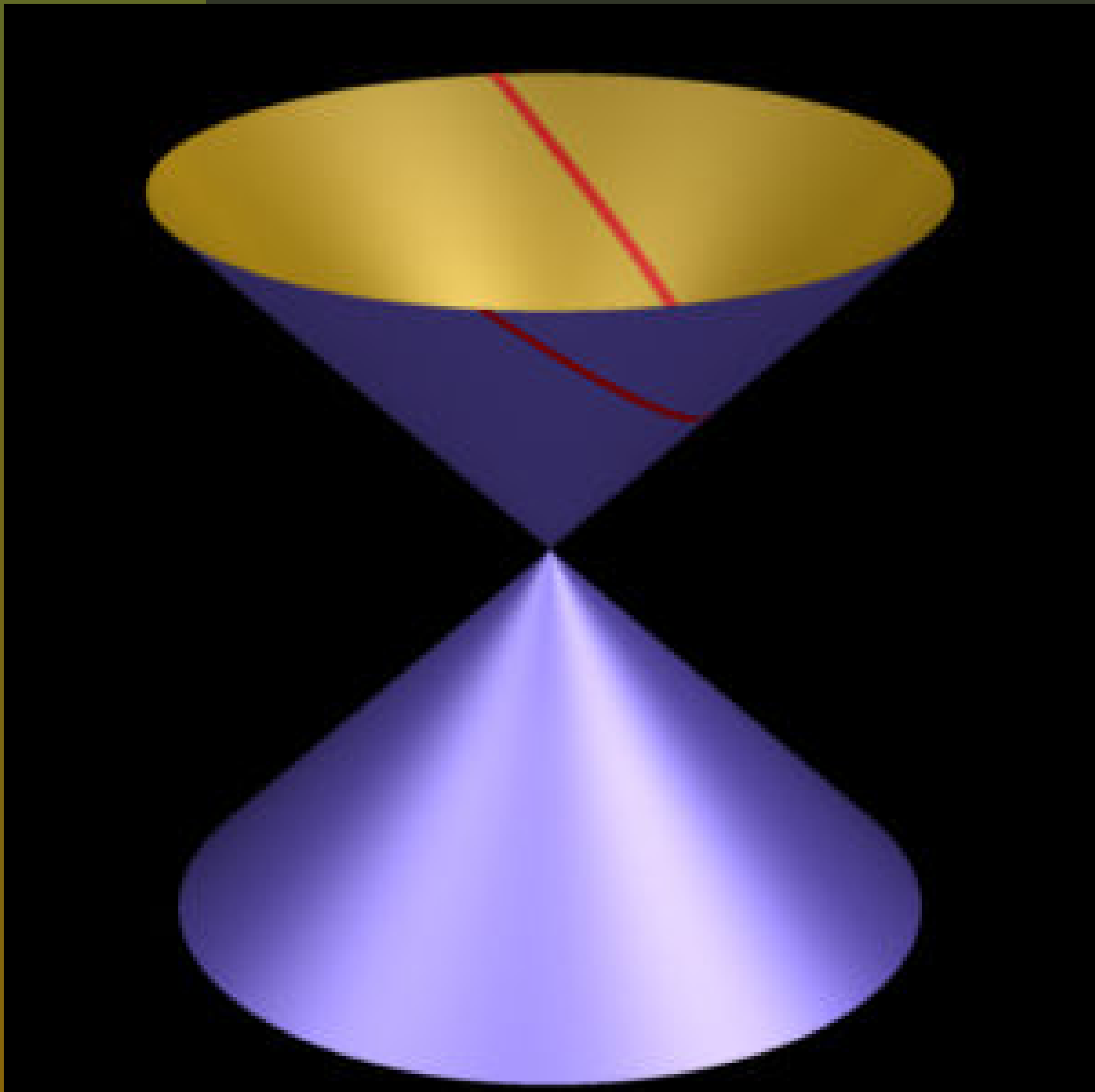
# Conic sections: ellipse



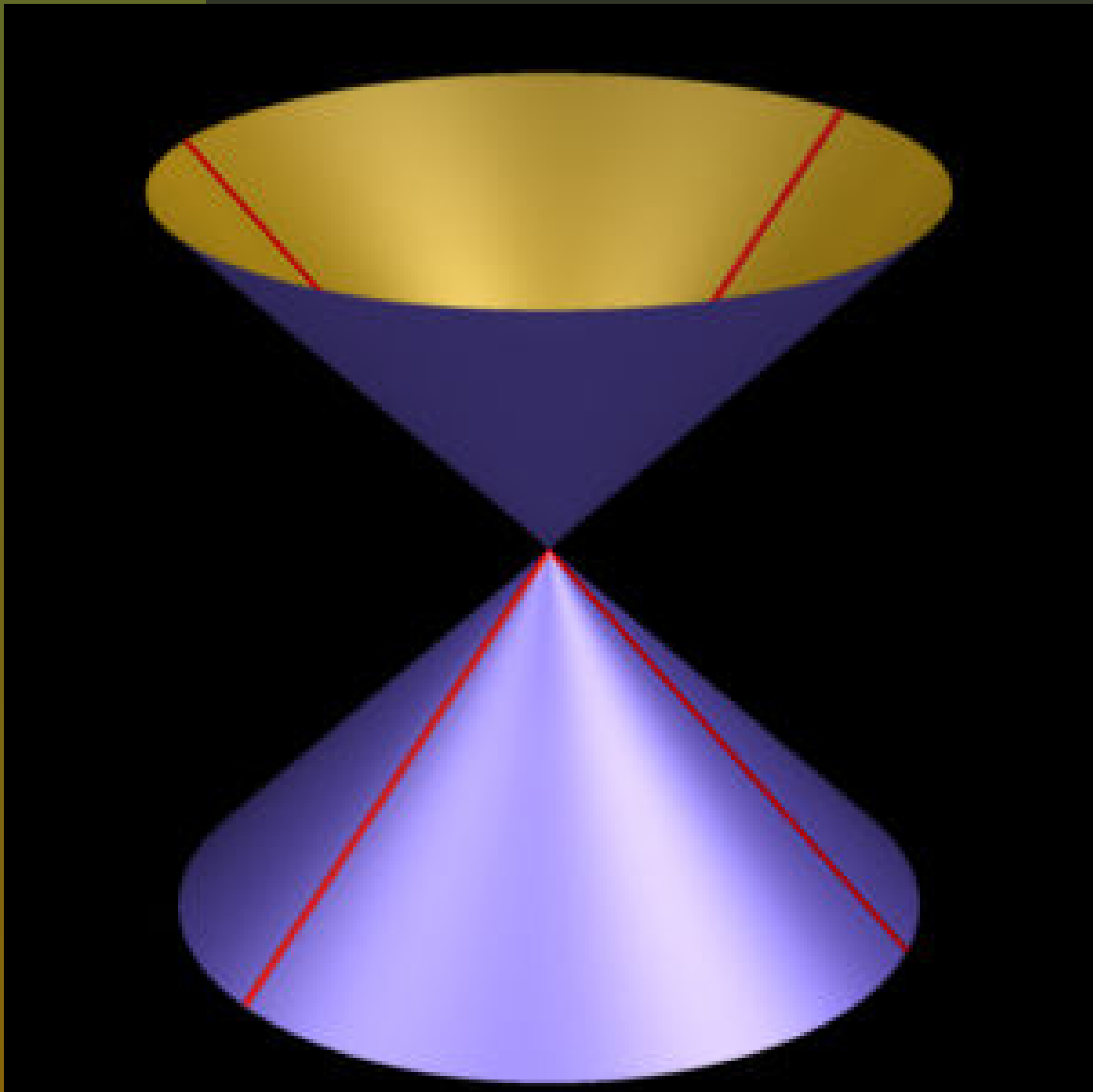
# Conic sections: hyperbola



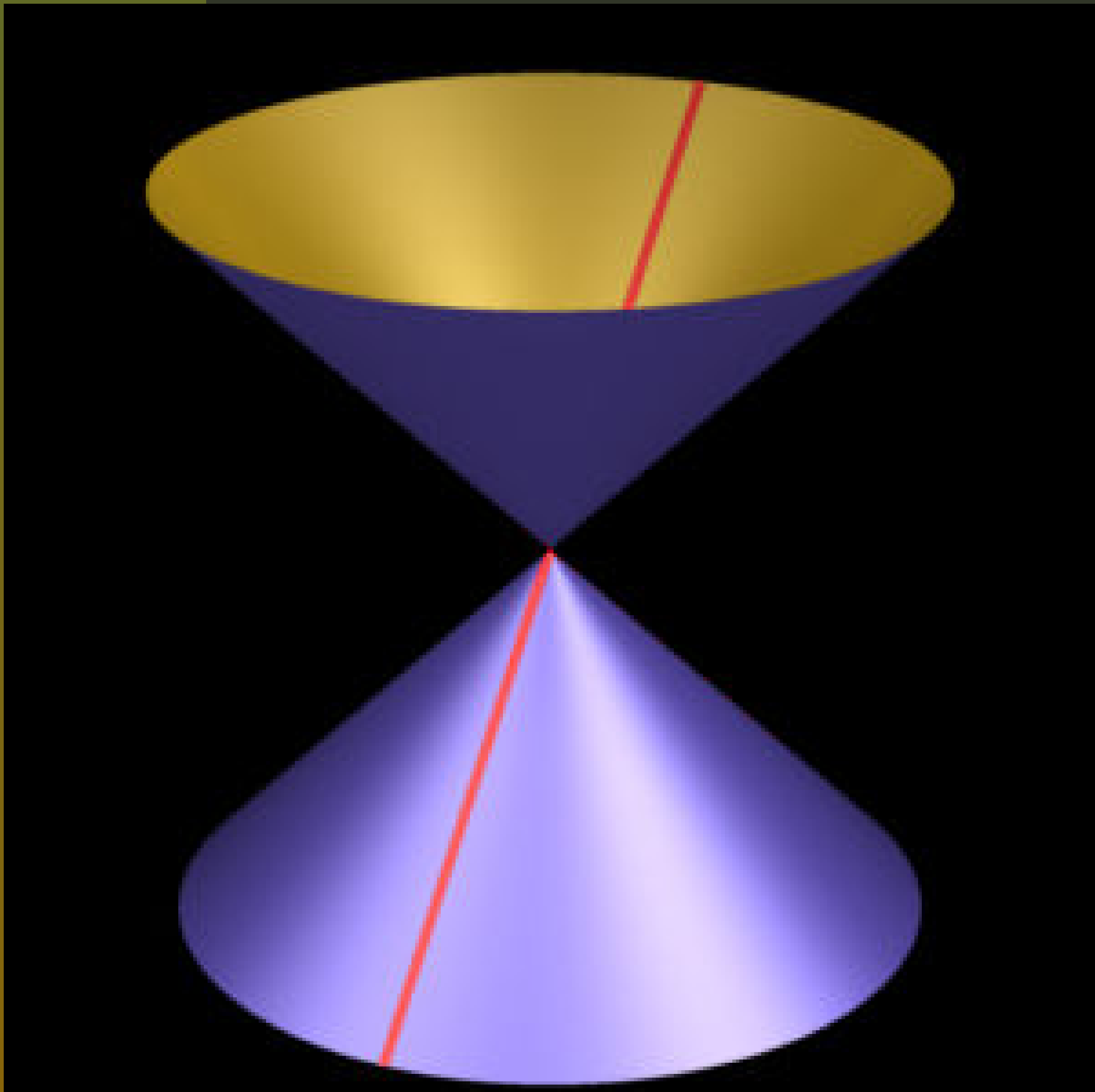
# Conic sections: parabola



# Conic sections: pair of lines



# Conic sections: double line



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- double line: $x^2 = 0$.
- The algebraic perspective both unifies and offers the opportunity to consider more complicated examples:

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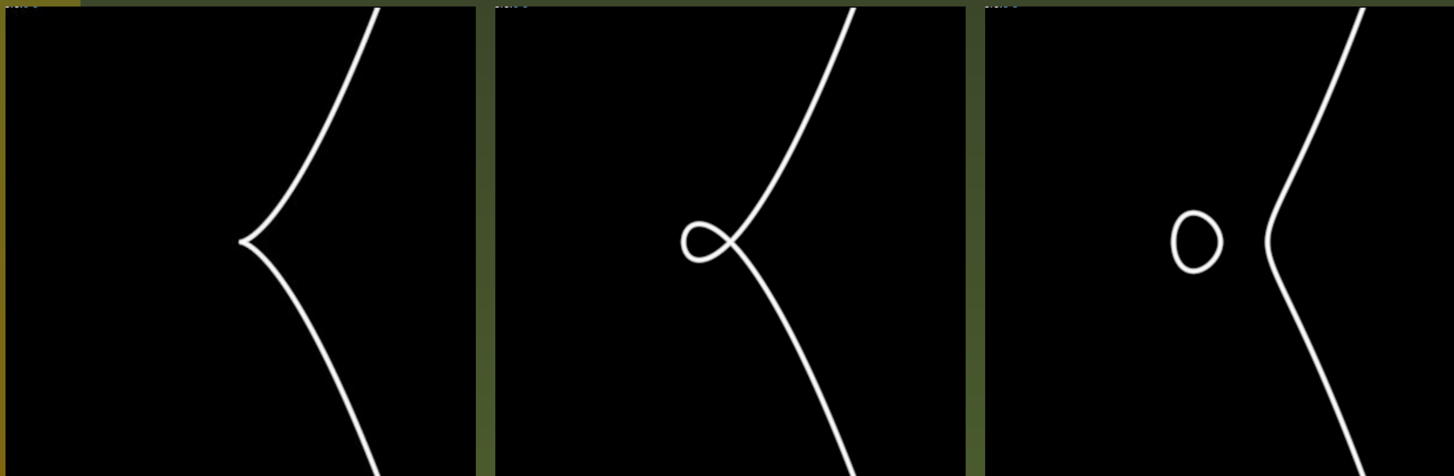

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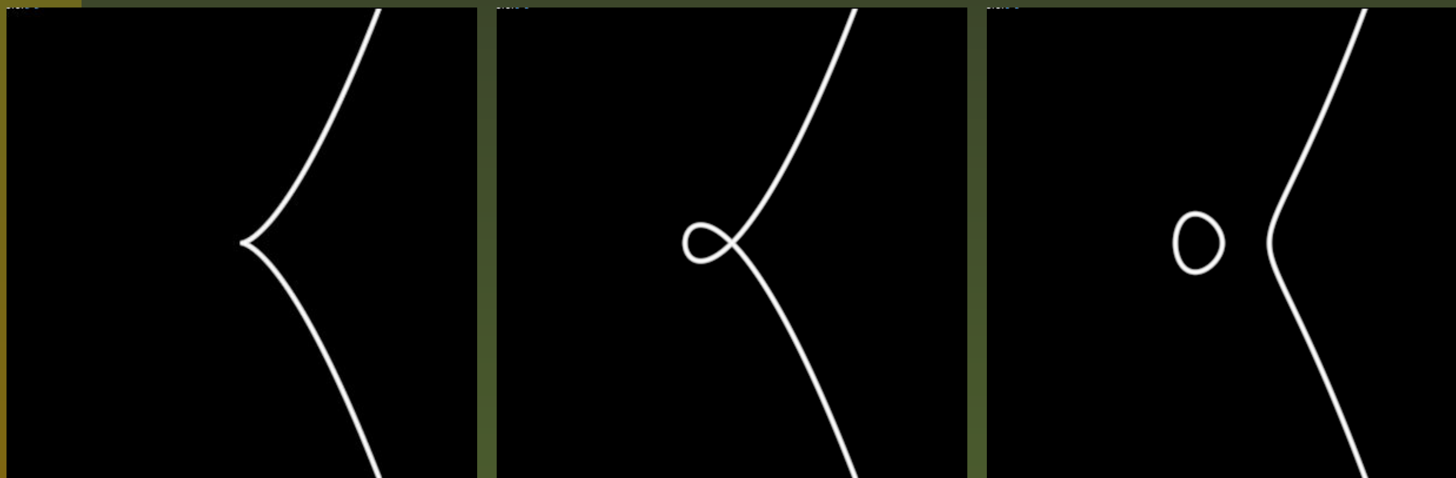

- Newton, circa 1700, looked at cubics and found 72 different (topological) types of cubics.

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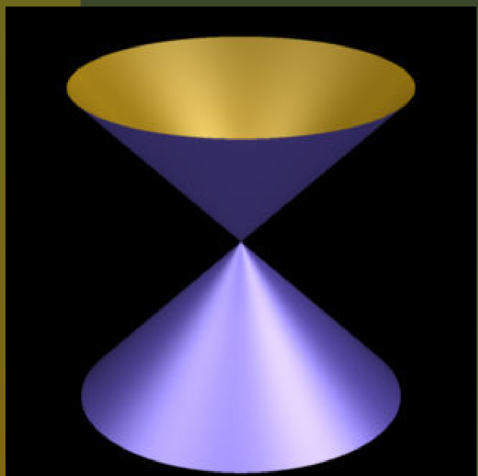  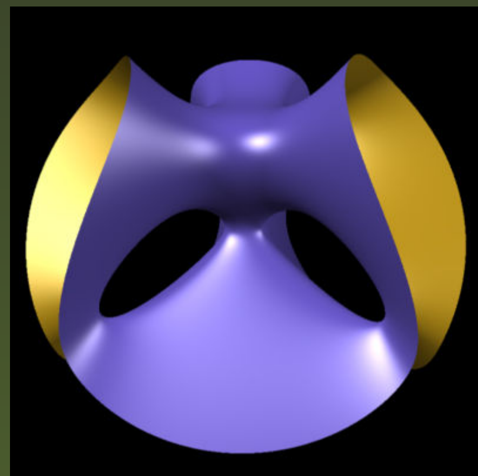

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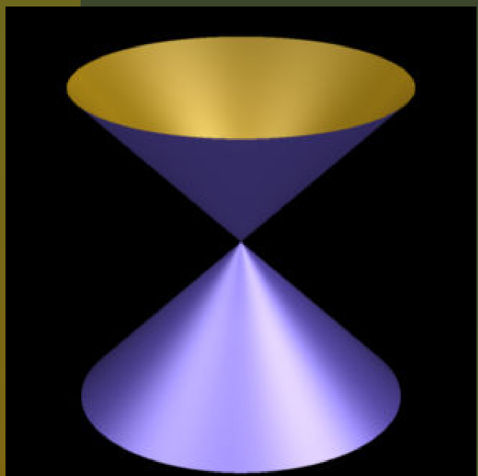
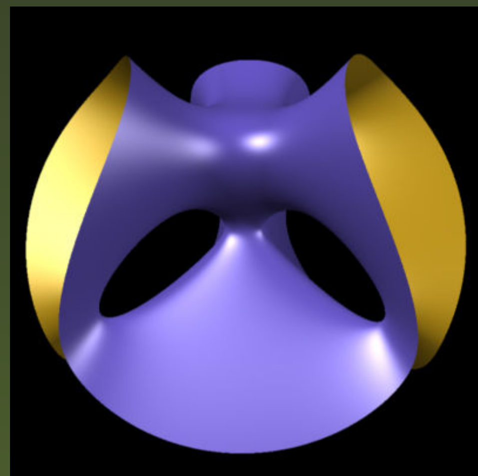
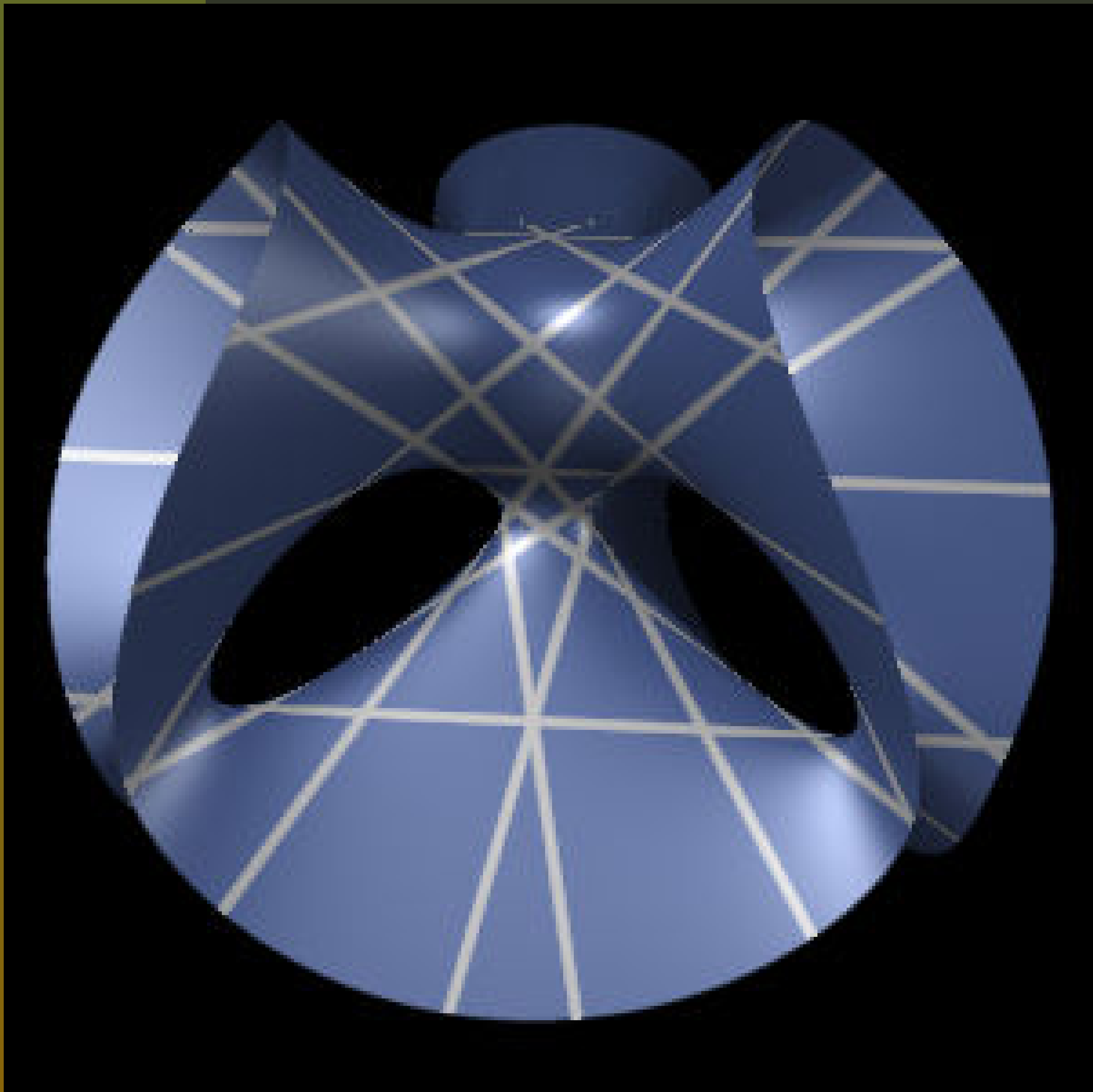




- Cayley communicated to the Royal society the discovery of the 27 lines on a cubic surface in 1869.

# 27 lines on the Clebsch cubic



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- Many of the interesting practical applications of algebraic geometry involve solving such systems.

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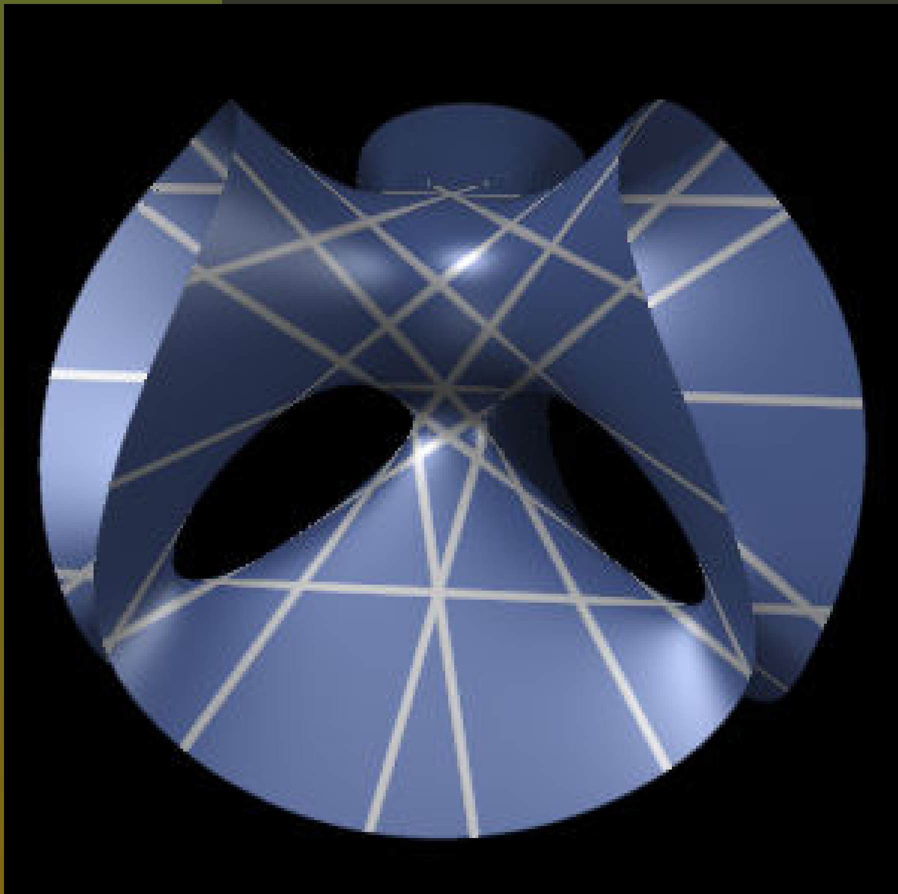
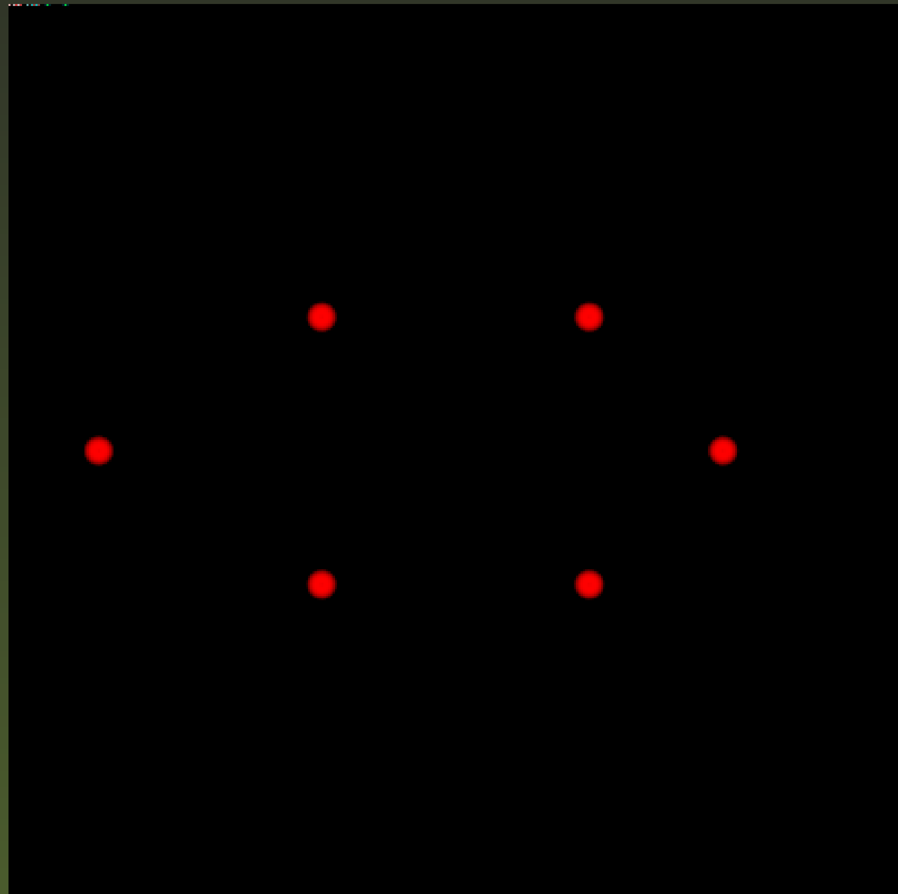
- The six skew lines, the fifteen lines connecting the six points we blow up and the six conics which pass through five of the six points.

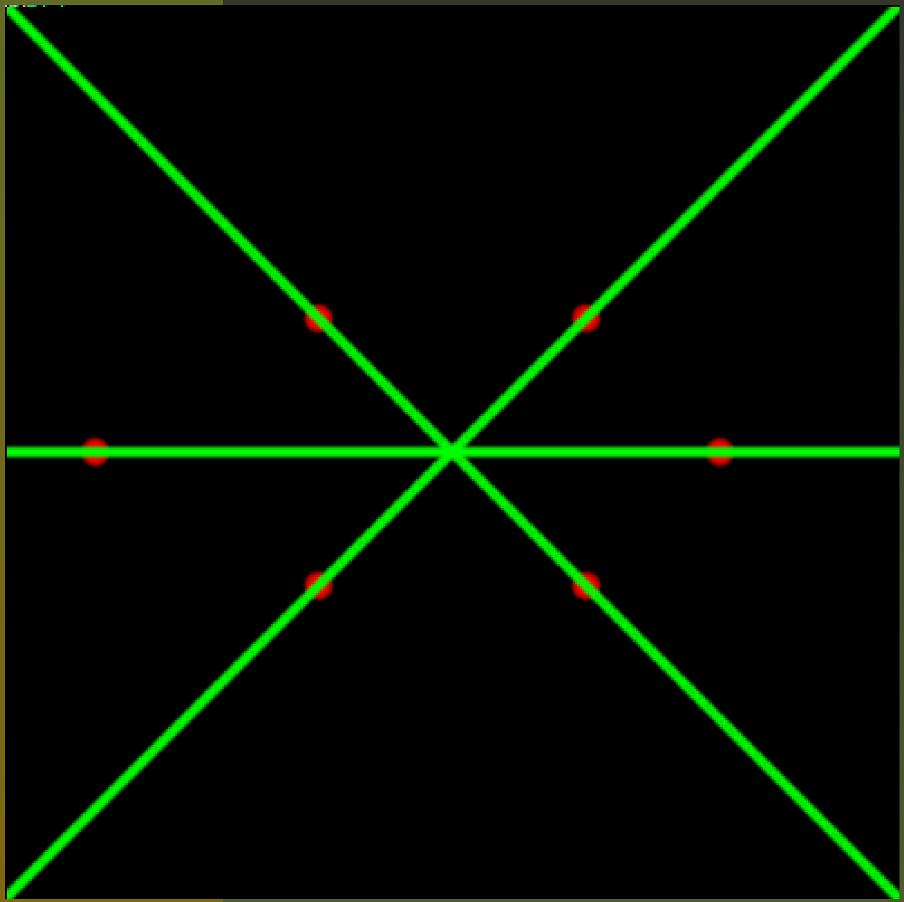
$$27 = 6 + 15 + 6.$$

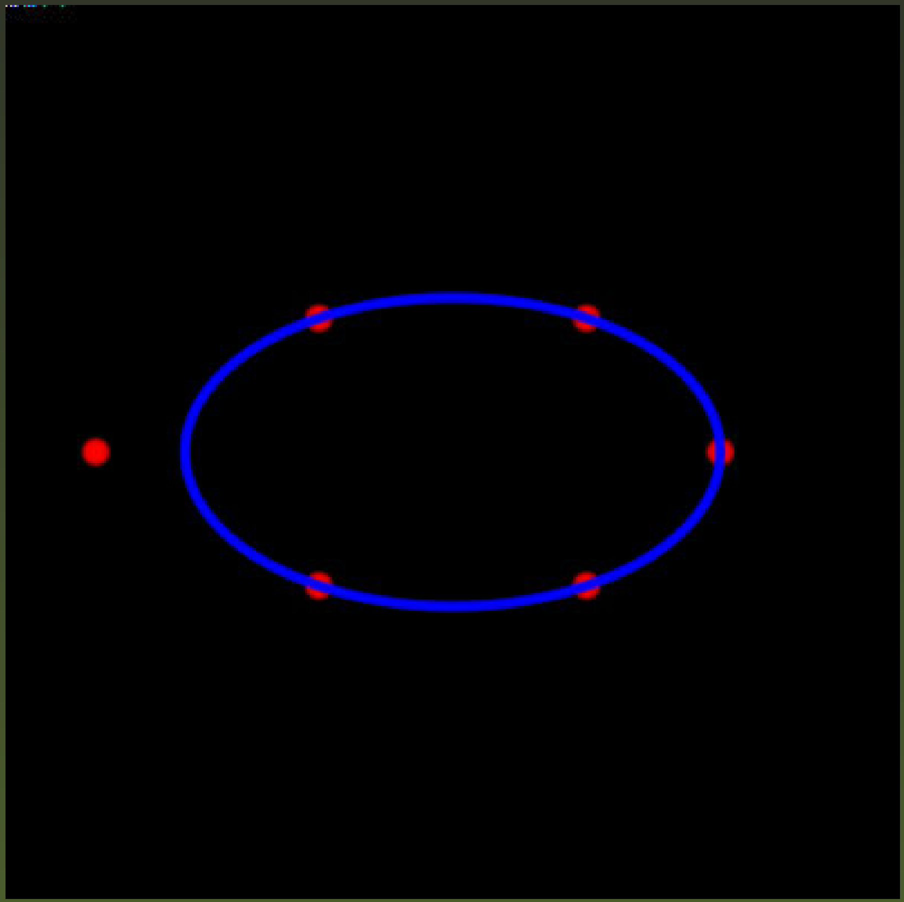
# Clebsch cubic, red lines





# Examples of green and blue lines





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- Closely related to flops are flips.

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- These slides were produced using latex (prosper) and surf (a program to draw curves and surfaces).