A new approach to Mori theory

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- The idea is to focus on three extreme cases,
  - K_X is ample.
  - K_X is trivial.
  - $-K_X$ is ample.
- The hope is that any variety is constructed, in a sense to be explained, using only these building blocks.

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$$p^*D = q^*D' + E,$$

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then $E \geq 0$, where both D and $D' = f_*D$ are $\mathbb{Q}$-Cartier.
- A log pair is a pair $(X, \Delta = \sum_i a_i\Delta_i)$, where X is normal, $\Delta \geq 0$ and $K_X + \Delta$ is $\mathbb{Q}$-Cartier, where $a_i \in [0, 1]$, for all i.

Minimal model Conjecture

Conjecture (Minimal Model Conjecture). *Let* (Y, Γ) *be a log smooth projective pair.*
Then there is a $K_Y + \Gamma$*-negative birational contraction* $f \colon Y \dashrightarrow X$ *and a morphism* $\pi \colon X \longrightarrow S$ *such that either*

1. $K_X + \Delta = \pi^* H$*, where* H *is ample, or*
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Note that this is the contraction of two conjectures, the minimal model conjecture and the abundance conjecture.

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- The problem is that both of these parts seem hard.

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- If π is divisorial replace (Y, Γ) by (X, Δ).
- If π is small, then $K_X + \Delta$ is not $\mathbb{Q}$-Cartier, and we need to do something different.

The flip of π

Instead we try to replace Y by another birational model Y^+, $Y \dashrightarrow Y^+$, such that $\pi^+\colon Y^+ \longrightarrow X$ is $(K_{Y^+} + \Gamma^+)$-ample.

$$\begin{tikzcd}
Y \arrow[rr, dashed, "\phi"] \arrow[dr, "\pi"'] & & Y^+ \arrow[dl, "\pi^+"] \\
& X. &
\end{tikzcd}$$

Three main Conjectures

Conjecture. *(Existence) Suppose that* $K_X + \Delta$ *is kawamata log terminal. Let* $f\colon X \longrightarrow Y$ *be a small extremal contraction.*
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Conjecture. *(Abundance) Suppose that* $K_X + \Delta$ *is kawamata log terminal and nef.*
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- Let $G_1, G_2, \dots, G_k$ be k $\mathbb{Q}$-divisors and let Y be a projective variety. The Cox ring is the multigraded ring

$$R(Y, G^\bullet) = \bigoplus_{m \in \mathbb{N}^k} H^0(Y, \mathcal{O}_Y(\llcorner \sum_i m_i G_i \lrcorner)).$$

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- If $R(Y, G)$ is finitely generated, where $G = K_Y + \Gamma$ ($k = 1$), then the log canonical model is equal to

$$X = \operatorname{Proj} R(Y, G).$$

One conjecture to rule them all

- **Conjecture (Finite Generation).** *Let* (Y, Γ_i) *be a log smooth pair, where* Γ_i *has rational coefficients, where* Y *is projective. Set* $G_i = K_Y + \Gamma_i$.
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- This conjecture does indeed imply the minimal model conjecture.
- There is some chance that attacking finite generation directly is better than the step by step approach sketched previously.

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- We will use this to run a special MMP.
- Suppose that $K_Y + \Gamma + tD$ is nef, for some divisor D and real number t (e.g. take D ample and t large).
- Assume that $K_Y + \Gamma + tD$ is kawamata log terminal and $K_Y + \Gamma$ is big.

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- At this point we use existence of flips, which is guaranteed by finite generation.

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- Then $-(K_X + \Delta)$ is relatively big. We may now modify π so that $-(K_X + \Delta)$ is ample, using the MMP with scaling.
- Further as f is $K_Y + \Gamma + tH$ negative, f is surely $K_Y + \Gamma$ negative.

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- $\kappa(K_X + \Delta) \geq 0$ and
- if $K_X + \Delta$ is not numerically trivial, then $\kappa(K_X + \Delta) = \kappa(K_Y + \Gamma) > 0$.

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- In addition $K_W + \Theta$ is big and log canonical.

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- Now suppose that $\kappa(K_X + \Delta) = 0$. X cannot be contracted by f to a point, since otherwise $K_X + \Delta$ is numerically trivial.
- So sections of $K'_W + \Theta'$ restricted to Z the image of Y give sections of $K_Y + \Delta$, and $\kappa(K_Y + \Gamma) > 0$.

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- Since we are assuming finite generation, (W, Θ) has a log canonical model (W', Θ'), $f \colon W \dashrightarrow W'$
- f is birational as $K_W + \Theta$ is big.
- First suppose that $\kappa(K_X + \Delta) = -\infty$. Then running the MMP with scaling, it follows that X is covered by curves C such that $(K_X + \Delta) \cdot C < 0$.
- Now suppose that $\kappa(K_X + \Delta) = 0$. X cannot be contracted by f to a point, since otherwise $K_X + \Delta$ is numerically trivial.
- So sections of $K'_W + \Theta'$ restricted to Z the image of Y give sections of $K_Y + \Delta$, and $\kappa(K_Y + \Gamma) > 0$.
- Thus finite generation implies abundance.

Finite Generation

- Baby case Suppose that $G = K_Y + \Gamma$ is very ample. Then considering

$$0 \longrightarrow \mathcal{O}_Y((k-1)G) \longrightarrow \mathcal{O}_Y(kG) \longrightarrow \mathcal{O}_G(kG) \longrightarrow 0,$$

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  and using Serre vanishing, we can prove by induction on the dimension that $R(X, G)$ is finitely generated.
- In general we cannot apply vanishing directly, since $K_Y + \Gamma$ is neither nef nor big.
- However the theory of multiplier ideals, developed by Siu and Kawamata, gives a way to lift sections (hopefully generators), from log canonical centres, under suitable assumptions.

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- Another idea of Shokurov gives a way to check finite generation locally.

Further work

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- As a baby case, consider the problem of giving an algebraic proof of Siu's deformation invariance of plurigenera.
- Does finite generation imply termination of flips in general?
- Does the MMP with scaling imply termination of flips in general?

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- Cascini and La Nave, math.AG/0603064, and indendently Tian and Zhang, already have some interesting results in this direction.
- They are able to exhibit some of the steps of the MMP using the methods of Ricci flow.