Birational classification of varieties

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- It is the aim of higher dimensional geometry to classify algebraic varieties up to birational equivalence.
- Thus the objects are algebraic varieties, but what are the morphisms?

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- Mori theory does not say much about finite maps.
- It does have a lot to say about morphisms with connected fibres.
- In fact any morphism $f\colon X \longrightarrow Y$ such that $f_*\mathcal{O}_X = \mathcal{O}_Y$ will be called a contraction morphism. If X and Y are normal, this is the same as requiring the fibres of f to be connected.

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- Traditionally the approved way to study a projective variety is to embed it in projective space, and consider the family of hyperplane sections.
- In Mori theory, we focus on curves, not divisors.
- In fact a contraction morphism $f\colon X \longrightarrow Y$ is determined by the curves which it contracts. Indeed Y is clearly determined topologically, and the condition $\mathcal{O}_Y = f_*\mathcal{O}_X$ determines the algebraic structure.

The closed cone of curves

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- Given f, set $D = f^*H$, where H is an ample divisor on Y. Then D is nef, that is $D \cdot C \geq 0$, for every curve C.

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- So, which faces F correspond to contractions f? Similarly which divisors are the pullback of ample divisors?
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- Note that if D is semiample, it is certainly nef.

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- The corresponding morphisms are the identity, the constant map to a point, and the two projections.
- In this example, the correspondence between faces and contractions is complete and in fact every nef divisor is semiample.

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- $\mathrm{Aut}(X)$ is large; it contains $\mathrm{SL}(2, \mathbb{Z})$.
- There are many contractions. Start with either of the two projections and act by $\mathrm{Aut}(X)$.

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- $\mathrm{NE}(X)$ is one half of the classic circular cone $x^2 + y^2 = z^2 \subset \mathbb{R}^3$. Thus many faces don't correspond to contractions.
- Many nef divisors are not semiample. Indeed, even on an elliptic curve there are numerically trivial divisors which are not torsion.

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- One contraction is given by the Abel-Jacobi map, and there is a similar map which contracts δ.
- But what happens when g and d are both large?

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- Indeed the existence of such a curve would imply that the pullback of S along $\Sigma \longrightarrow C$ splits, which contradicts stability.
- We really need to take the closure, to defi ne $\mathrm{NE}(S)$.

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- What went wrong?

The canonical divisor

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- The basic moral is that the cone of curves is nice on the negative side, and that if we contract these curves, we get a reasonable model.
- Consider the case of curves.

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- So let us now consider surfaces.

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  - $S \longrightarrow C$, where K_S is zero on the fibres. If C is a curve, the fibres are elliptic curves.
  - K_S is ample. S is of general type. Note that S is forced to be singular in general.
- The problem, as we have already seen, is that we can destroy this picture, simply by blowing up. It is the aim of the MMP to reverse the process of blowing up.

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- In the second case there is a K_X-extremal ray R. That is to say R is extremal in the sense of convex geometry, and $K_X \cdot R < 0$.
- Moreover, we can contract R, $\phi_R \colon X \longrightarrow Y$.

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  - Z is a curve. The fi bres are copies of $\mathbb{P}^1$.
  - Z is a surface. ϕ blows down a -1-curve.

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- If $\dim Z < 2$, then STOP.
- If $\dim Z = 2$ then replace S with Z, and continue.

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- By the Cone Theorem, there is an extremal contraction, $\pi\colon X \longrightarrow Y$, of relative Picard number one such that for a curve C', $\pi(C')$ is a point iff C' is homologous to a multiple of C.

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- If π is birational and the locus contracted by π is a divisor, then even though Y might be singular, it will at least be $\mathbb{Q}$-factorial (for every Weil divisor D, some multiple is Cartier).
  Replace X by Y and keep going.

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- If the locus contracted by π is not a divisor, that is, π is small, then Y is not $\mathbb{Q}$-factorial.
- Instead of contracting C, we try to replace X by another birational model X^+, $X \dashrightarrow X^+$, such that $\pi^+ \colon X^+ \longrightarrow Y$ is K_{X^+}-ample.

$$\begin{tikzcd}
X \arrow[rr, dashed, "\phi"] \arrow[dr, "\pi"'] & & X^+ \arrow[dl, "\pi^+"] \\
& Z. &
\end{tikzcd}$$

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- Even supposing we can perform a flip, how do know that this process terminates?
- It is clear that we cannot keep contracting divisors, but why could there not be an infinite sequence of flips?

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- Both of these results have far reaching generalisations, whose form dictates the main definitions of the subject.

An illustrative example

- Let S be a smooth projective surface and let $E \subset S$ be a -1-curve, that is $K_S \cdot E = -1$ and $E^2 = -1$. We want to contract E.

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- Then pick b so that $(K_S + aH + bE) \cdot E = 0$. Note that $b > 0$ (in fact typically b is very large).
- Now we consider the rational map given by $|mD|$, for $m >> 0$ and suffi ciently divisible.

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- So by Kawamata-Viehweg Vanishing

$$H^1(S, \mathcal{O}_S(mD - E)) = H^1(S, \mathcal{O}_S(K_S + G + (m-1)D)) = 0$$

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- B is ample, so we have the start of an induction.
- By vanishing, the map

$$H^0(S, \mathcal{O}_S(mD)) \longrightarrow H^0(E, \mathcal{O}_E(mD))$$

is surjective. Thus $|mD|$ is base point free and the resulting map $S \longrightarrow T$ contracts E.

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- Observe that if we set $G' = \pi_* G$, then G' has high multiplicity along p, the image of E (that is b is large).
- In general, we manufacture a divisor E by picking a point $x \in X$ and then pick H with high multiplicity at x.
- Next resolve singularities $\tilde{X} \longrightarrow X$ and restrict to an exceptional divisor E, whose centre has high multiplicity w.r.t H (strictly speaking a log canonical centre of $K_X + H$).

Singularities in the MMP

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- We say that the pair (X, Δ) is klt if the coeffi cients of Γ are always less than one.

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- Moreover if $K_X + S + B$ is plt then $K_S + D$ is klt.

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- If we take a cover with appropriate ramifi cation, then we can eliminate any component with coeffi cient less than one.
- (Kawamata-Viehweg vanishing) Suppose that $K_X + \Delta$ is klt and L is a line bundle such that $L - (K_X + \Delta)$ is big and nef. Then, for $i > 0$,

$$H^i(X, L) = 0.$$

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- To achieve this birational classification, we propose to use the MMP.

Two main Conjectures

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Conjecture. *(Existence) Suppose that $K_X + \Delta$ is kawamata log terminal. Let $\pi\colon X \longrightarrow Y$ be a small extremal contraction.*
Then the flip of π exists.

Conjecture. *(Termination) There is no infinite sequence of kawamata log terminal flips.*

Abundance

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Conjecture. *(Abundance) Suppose that $K_X + \Delta$ is kawamata log terminal and nef.*
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Considering the resulting morphism $\phi\colon X \longrightarrow Y$, we recover the Kodaira-Enriques classifi cation of surfaces.