An invitation to log geometry

James M$^{\text{C}}$Kernan

UCSB

An easy integral

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so we get $\pi/2$.

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Let us look more carefully at the basic integral.

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- Put differently if we denote the above integral by u, then $t = \sin(u)$.
- Note that $\sin$ is a more interesting function than $\sin^{-1}$.
- Indeed it is periodic; has an interesting additivity property and it extends to the whole complex plane.

A simple addition formula

Consider computing c in terms of a and b, where

$$\int_0^a \frac{dx}{\sqrt{1-x^2}} + \int_0^b \frac{dx}{\sqrt{1-x^2}} = \int_0^c \frac{dx}{\sqrt{1-x^2}}.$$

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Thus

$$\begin{aligned} c &= \sin(\gamma) \\ &= \sin(\alpha+\beta) \\ &= \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) = a(1-b^2)^{1/2} + b(1-a^2)^{1/2}. \end{aligned}$$

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- Define a function $\phi(z)$ by the rule

$$z = \int_0^{\phi(z)} \frac{dx}{\sqrt{f(x)}},$$

  where z is a complex number.
- Any such function ϕ is called an elliptic function.

Properties of elliptic functions

- $\phi(z)$ is doubly periodic, that is

$$\phi(z + m\omega_1 + n\omega_2) = \phi(z),$$

for two complex numbers ω_1 and ω_2 which are independent over $\mathbb{R}$.

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- $\phi(z)$ satisfies an addition formula similar to the easy case.
- Specifically if we define z_3 as a function of z_1 and z_2 by the formula

  $$\phi(z_1) + \phi(z_2) = \phi(z_3),$$

  then z_3 is a rational function of z_1 and z_2.

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- Let us look at the problem from two different perspectives.
- From an algebraic perspective, we are really looking at algebraic curves

$$y = \sqrt{f(x)} \qquad \text{so that} \qquad y^2 = f(x).$$

Algebraic Perspective

- If $f(x) = 1 - x^2$, then we can parametrise the curve. Indeed if we project from the point $(0, 1)$ down to the x-axis, so that the point (x, y) projects down to $(t, 0)$, then we obtain the standard parametrisation

$$x = \frac{2t}{1+t^2} \qquad \text{and} \qquad y = \frac{t^2-1}{1+t^2}.$$

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  Similarly for any quadratic polynomial.
- The explains why the first integral is so easy.
- It turns out if $f(x)$ is a general cubic (has no repeated roots) then there is no such parametrisation. To see this we need the second perpsective.

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- Thus as a real manifold, our algebraic curve is a real surface.
- In fact it is easy to compactify (passing from affine space $\mathbb{A}^2 = \mathbb{C}^2$ to projective space $\mathbb{P}^2$) so that we have a compact Riemann surface.

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- The second surface is isomorphic to an elliptic curve, a quotient of $\mathbb{C}$ by a two dimensional lattice $\Lambda \simeq \mathbb{Z}^2$.
- The periods ω_1 and ω_2 are generators for the Λ, so that $\phi(z)$ descends to the elliptic curve.
- It turns out that there is no holomorphic map (let alone algebraic) from the Riemann sphere to an elliptic curve. To see this we need differential forms.

Differential Forms

- To get a differential form, just drop the integral sign from:

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- Differential forms transform via the Jacobian matrix.
- Given a complex manifold (or algebraic variety), the most important invariant of a form ω is its zero locus and polar locus, which is a divisor.

Divisors

- A divisor D is a formal linear combination of codimension one subvarieties:

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- Given a differential form ω, locally it is of the form $f(z)dz_1 \wedge dz_2 \wedge dz_3 \ldots dz_n$ and we take the zeroes minus the poles of f

$$(f)_0 - (f)_\infty,$$

to get a divisor K_X, which we call the canonical divisor.

Curves

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- Riemann proved that if the genus of C is g, then the canonical divisor has degree $2g-2$.
- Moreover if $f\colon C \longrightarrow B$ is a non-constant holomorphic map of Riemann surfaces then

$$2g-2 = d(2h-2) + b,$$

where $b \geq 0$ count the number of branch points.

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- Then $g = 0$ and $2g - 2 = -2$.
- Thus
$$-2 = d(2h-2) + b,$$
which forces 0. But then $B \simeq \mathbb{P}^1$ and $\mathbb{P}^1$ never covers an elliptic curve, which has genus one.

Behaviour of the canonical divisor

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- Let U be a quasi-projective variety. Then by Hironaka we may embed U into a projective variety X, such that the complement is a divisor D, such that the pair (X, D) is smooth.
- It turns out that the divisor $K_X + D$ reflects the geometry.

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