

Finite generation of canonical rings I

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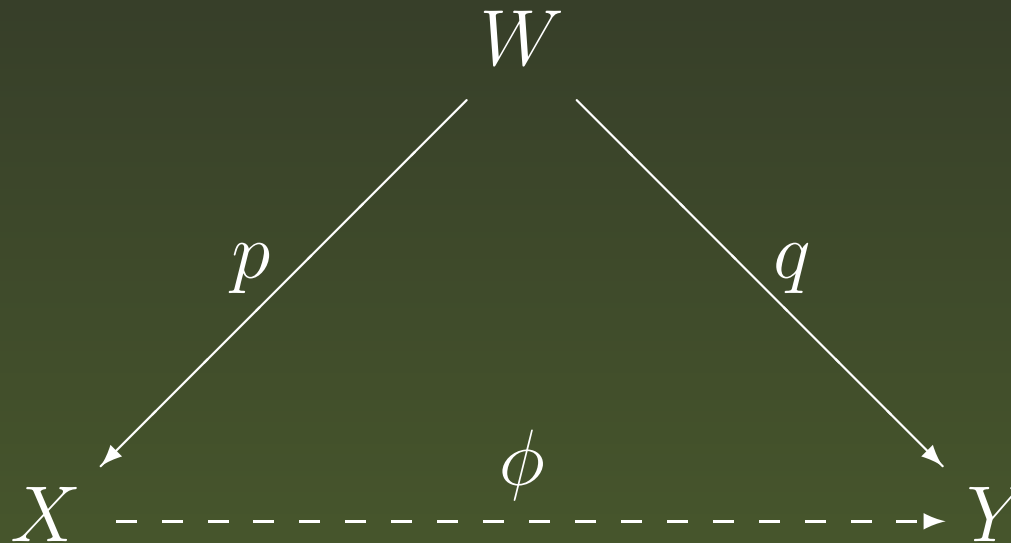
Definition. *Let $\pi: X \rightarrow U$ be a projective morphism of normal varieties, and let $\phi: X \dashrightarrow Y$ be a birational map over U , whose inverse does not contract any divisors. Let D be an \mathbb{R} -Cartier divisor on X .*

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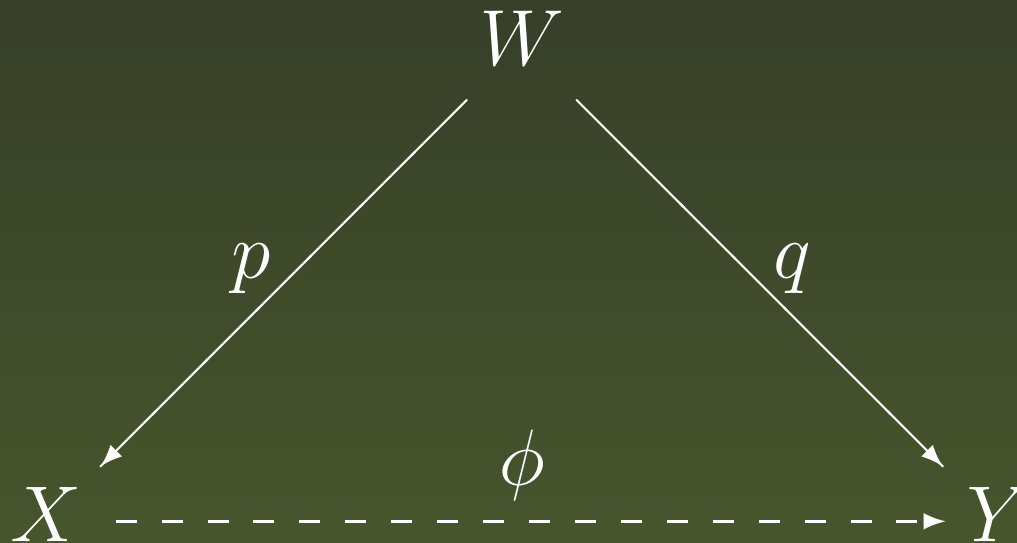
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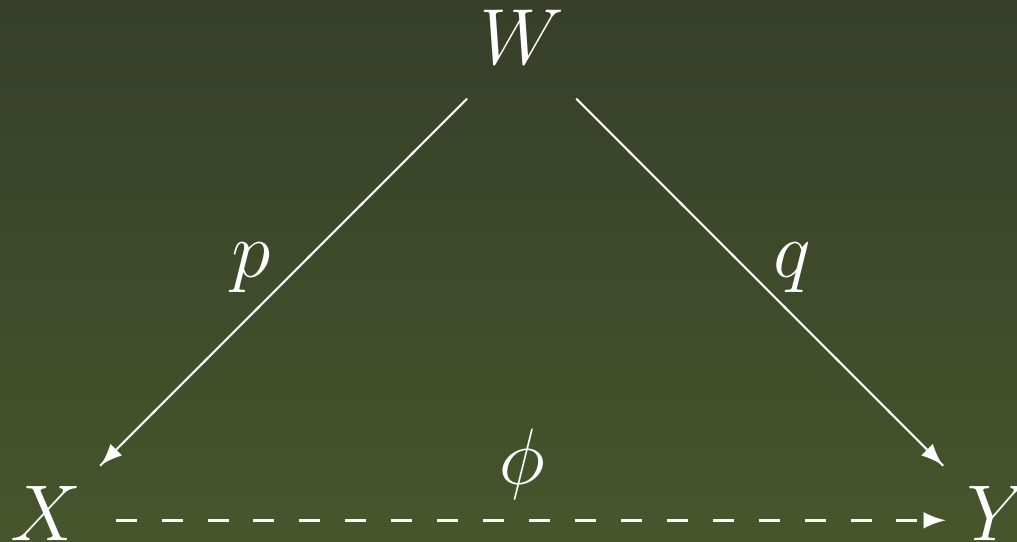
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$p^* D = q^* D' + E$, where $E \geq 0$ E contains the transform of every divisor exceptional for $X \dashrightarrow Y$. Key Point:

$$H^0(X, \mathcal{O}_X(\lfloor mD \rfloor)) \simeq H^0(Y, \mathcal{O}_Y(\lfloor mD' \rfloor)), \quad \forall m \geq 0.$$

Log terminal model

Definition. Let $\pi: X \longrightarrow U$ be a proper morphism of normal varieties. Let (X, Δ) be a kawamata log terminal pair. A **log terminal model** for $K_X + \Delta$ over U is a $(K_X + \Delta)$ -negative rational map $\phi: X \dashrightarrow Y$ over U , where $Y \longrightarrow U$ is projective, Y is \mathbb{Q} -factorial and $K_Y + \Gamma = K_Y + \phi_*\Delta$ is nef.

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Theorem. [Birkar, Cascini, Hacon, -] Let $\pi: X \longrightarrow U$ be a proper morphism of normal varieties. If $K_X + \Delta$ is π -pseudo-effective and Δ is big over U , then $K_X + \Delta$ has a log terminal model over U .

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I would like to spend most of the rest of the talk drawing some conclusions from the main theorem.

Minimal and Canonical models

Corollary. *[Birkar, Cascini, Hacon, -; Siu] Let X be a smooth projective variety of general type. Then*

1. *X has a minimal model,*
2. *X has a canonical model.*

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A similar proof works when $K_X + \Delta$ is big and kawamata log terminal.

Finite generation

Corollary. *[Birkar, Cascini, Hacon, -; Siu] Let X be a smooth projective variety. Then the canonical ring*

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Proof. Fujino and Mori proved that if $f : X \dashrightarrow Y$ is the Iitaka fibration then

$$R(X, K_X) = R(Y, K_Y + \Gamma) = \bigoplus_{m \in \mathbb{N}} H^0(Y, \mathcal{O}_Y(\lfloor m(K_Y + \Gamma) \rfloor)).$$

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To see what sort of MMP we can run, it will be useful to state a rough version of part of the induction to prove the main theorem:

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The set

$\{ Y \mid (Y, \Gamma) \text{ is a log terminal model of } K_X + A + \Delta, \text{ where } \Delta = \sum a_i \Delta_i, K_X + A + \Delta \text{ is kawamata log terminal} \},$
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For the induction, we will need a version of this result in the relative setting.

- Start with a log smooth pair (X, Δ) , where X is pseudo-effective and a divisor H such that $K_X + \Delta + H$ is nef.

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- π is divisorial. Replace X by Z and return.
- π is small. Replace X by the flip and return.

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- Thus the MMP with scaling terminates by Theorem C.

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Corollary. *Let X be a smooth projective variety. If K_X is not pseudo-effective then there is a K_X -negative map $\phi: X \dashrightarrow Y$ and $f: Y \longrightarrow Z$ a Mori fibre space.*

Mori fibre spaces

Proof. Pick H very ample smooth such that $K_X + H$ is ample.

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Then $0 < c < 1$, so that $K_X + \Delta = K_X + cH$ is kawamata log terminal, and $K_X + \Delta$ is not big. Let $\phi: X \dashrightarrow Y$ be a log terminal model. Then ϕ is K_X -negative as it is $(K_X + \Delta)$ -negative and Δ is ample.

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Algebraic/Moishezon spaces

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- Then at both points we have a choice as to which curve to blow up first.
- In the category of algebraic spaces, we can find a morphism $f : Y \longrightarrow X$, which realises different choices at x and y .
- It is easy to see that Y is not a projective variety.
- Is this always the case? Ie if Y is an algebraic space which is not projective, then does Y have to contain a rational curve?

Rational curves on algebraic spaces

Theorem. *Let $\pi: X \longrightarrow U$ be a proper morphism of normal algebraic spaces. Suppose that $K_X + \Delta$ is kawamata log terminal and X is analytically \mathbb{Q} -factorial.*

If X does not contain any rational curves contracted by π , then π is a log terminal model.

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We prove this in the special case that U is a point.

Proof

Let $f: Y \longrightarrow X$ be a projective resolution of X . We may write

$$K_Y + \Gamma' = f^*(K_X + \Delta) + E.$$

Let $\Gamma = \Gamma' + \epsilon F$, where F is the full exceptional locus.

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So we may assume that $K_Y + \Gamma$ is nef. But then $E + \epsilon F$ must be empty. Thus $Y \longrightarrow X$ is small. As X is \mathbb{Q} -factorial, it follows that $Y = X$.

Theorem A, B and C

Theorem. A *Let X be a projective variety.*

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Theorem. B *Let X be a projective variety.*

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Theorem. C *Let X be a projective variety. Fix an ample divisor A . Fix divisors $\Delta_1, \Delta_2, \dots, \Delta_k$.*

The following set is finite:

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- A_n implies B_n . The proof mirrors Shokurov's non-vanishing result.
- A_n and B_n imply C_n . Use the ideas from Shokurov's threefold log models paper. One key point is that since we work with \mathbb{R} -divisors we can apply compactness.
- Finally A_n , B_n and C_n imply existence of flips in dimension $n + 1$. This uses ideas from Siu and Kawamata on lifting sections and Shokurov on saturated algebras.