Finite generation of canonical rings I

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Find a nice birational representative Y of X.

Definition. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal varieties, and let $\phi\colon X \dashrightarrow Y$ be a birational map over U, whose inverse does not contract any divisors. Let D be an $\mathbb{R}$-Cartier divisor on X.*

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$p^* D = q^* D' + E$, where $E \geq 0$ E contains the transform of every divisor exceptional for $X \dashrightarrow Y$. Key Point:

$$H^0(X, \mathcal{O}_X(\llcorner mD \lrcorner)) \simeq H^0(Y, \mathcal{O}_Y(\llcorner mD' \lrcorner)), \ \forall m \geq 0.$$

Log terminal model

Definition. *Let* $\pi: X \longrightarrow U$ *be a proper morphism of normal varieties. Let* (X, Δ) *be a kawamata log terminal pair. A* *log terminal model* *for* $K_X + \Delta$ *over* U *is a* $(K_X + \Delta)$*-negative rational map* $\phi: X \dashrightarrow Y$ *over* U*, where* $Y \longrightarrow U$ *is projective,* Y *is* $\mathbb{Q}$*-factorial and* $K_Y + \Gamma = K_Y + \phi_* \Delta$ *is nef.*

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Theorem. *[Birkar, Cascini, Hacon, -] Let* $\pi : X \longrightarrow U$ *be a proper morphism of normal varieties.*
If $K_X + \Delta$ *is* π*-pseudo-effective and* Δ *is big over* U*, then* $K_X + \Delta$ *has a log terminal model over* U*.*

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I would like to spend most of the rest of the talk drawing some conclusions from the main theorem.

Minimal and Canonical models

Corollary. *[Birkar, Cascini, Hacon, -; Siu] Let X be a smooth projective variety of general type. Then*

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A similar proof works when $K_X + \Delta$ is big and kawamata log terminal.

Finite generation

Corollary. *[Birkar, Cascini, Hacon, -; Siu] Let X be a smooth projective variety. Then the canonical ring*

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Proof. Fujino and Mori proved that if $f\colon X \dashrightarrow Y$ is the Iitaka fibration then

$$R(X, K_X) = R(Y, K_Y + \Gamma) = \bigoplus_{m \in \mathbb{N}} H^0(Y, \mathcal{O}_Y(\llcorner m(K_Y + \Gamma) \lrcorner)).$$

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where $K_Y + \Gamma$ is kawamata log terminal and big. But then (Y, Γ) has a log canonical model.

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To see what sort of MMP we can run, it will be useful to state a rough version of part of the induction to prove the main theorem:

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The set

$$\{\, Y \mid (Y, \Gamma) \text{ is a log terminal model of } K_X + A + \Delta \text{, where } \Delta = \sum a_i \Delta_i,\ K_X + A + \Delta \text{ is kawamata log terminal} \,\},$$

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For the induction, we will need a version of this result in the relative setting.

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- Thus the MMP with scaling terminates by Theorem C.

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We can also say something if K_X is not pseudo-effective:

Corollary. *Let X be a smooth projective variety. If K_X is not pseudo-effective then there is a K_X-negative map $\phi\colon X \dashrightarrow Y$ and $f: Y \longrightarrow Z$ a Mori fibre space.*

Mori fibre spaces

Proof. Pick H very ample smooth such that $K_X + H$ is ample.
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$$c = \inf\{\, t \in \mathbb{R} \mid K_X + tH \text{ is pseudo-effective} \,\}.$$

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- Then at both points we have a choice as to which curve to blow up first.
- In the category of algebraic spaces, we can find a morphism $f\colon Y \longrightarrow X$, which realises different choices at x and y.
- It is easy to see that Y is not a projective variety.
- Is this always the case? Ie if Y is an algebraic space which is not projective, then does Y have to contain a rational curve?

Rational curves on algebraic spaces

Theorem. *Let* $\pi\colon X \longrightarrow U$ *be a proper morphism of normal algebraic spaces. Suppose that* $K_X + \Delta$ *is kawamata log terminal and* X *is analytically* $\mathbb{Q}$*-factorial.*
If X *does not contain any rational curves contracted by* π*, then* π *is a log terminal model.*

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We prove this in the special case that U is a point.

Proof

Let $f : Y \longrightarrow X$ be a projective resolution of X. We may write

$$K_Y + \Gamma' = f^*(K_X + \Delta) + E.$$

Let $\Gamma = \Gamma' + \epsilon F$, where F is the full exceptional locus.

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Since X does not contain any rational curves, this is automatically a MMP over X. As Γ is big over X this MMP must terminate (note that termination may be checked locally in the étale topology.
So we may assume that $K_Y + \Gamma$ is nef. But then $E + \epsilon F$ must be empty. Thus $Y \longrightarrow X$ is small. As X is $\mathbb{Q}$-factorial, it follows that $Y = X$.

Theorem A, B and C

Theorem. *A Let X be a projective variety.*
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Theorem. *B* *Let X be a projective variety. If $K_X + \Delta$ is kawamata log terminal, Δ is big and $K_X + \Delta$ is pseudo-effective, then $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$.*

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Theorem. *C Let X be a projective variety. Fix an ample divisor A. Fix divisors $\Delta_1, \Delta_2, \dots, \Delta_k$. The following set is finite:*

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- A_n implies B_n. The proof mirrors Shokurov's non-vanishing result.
- A_n and B_n imply C_n. Use the ideas from Shokurov's threefold log models paper. One key point is that since we work with $\mathbb{R}$-divisors we can apply compactness.
- Finally A_n, B_n and C_n imply existence of flips in dimension $n+1$. This uses ideas from Siu and Kawamata on lifting sections and Shokurov on saturated algebras.